

Question 3

Consider the normalized Neumann problem

$$-u'' = x - \pi/2, \quad u'(0) = u'(\pi) = 0, \quad \int_0^\pi u(x)dx = 0.$$

Use the convention $u = a_0/2 + \sum_{n \geq 1} a_n \cos nx$.

Tasks

1. Derive the integral compatibility condition for a general forcing f with these derivative conditions. Explain why replacing the forcing by x makes the problem impossible.
2. Find the cosine coefficients of the stated forcing. Explain how the zero mode differs from every positive-frequency mode when solving for u .
3. Solve in closed form and as a cosine series. Verify the equation, both derivative conditions and the mean.
4. Prove uniqueness with the normalization, and state the full family without it. Explain why adding a constant cannot repair an incompatible forcing.

Question 3 – Solution

Strategy. Check the forcing mean before dividing by eigenvalues; the constant mode has eigenvalue zero.

Step 1: Check compatibility before solving. Integration gives $\int_0^\pi f = -u'(\pi) + u'(0) = 0$. The stated forcing satisfies this condition, whereas $\int_0^\pi x = \pi^2/2 \neq 0$. The replacement is impossible regardless of the additive level of u .

Step 2: Solve the mode equations. The forcing has mean zero and coefficients $F_n = 2((-1)^n - 1)/(\pi n^2)$. Since $-(\cos nx)'' = n^2 \cos nx$,

$$a_n = \frac{2((-1)^n - 1)}{\pi n^4} \quad (n \geq 1).$$

The zero-mode equation is $0 = 0$ and leaves the mean free. The normalization sets $a_0 = 0$; dividing this equation by zero would be invalid.

Step 3: Verify an independent polynomial solution. Direct integration gives

$$u = -\frac{x^3}{6} + \frac{\pi x^2}{4} - \frac{\pi^3}{24} = -\frac{4}{\pi} \sum_{j=0}^{\infty} \frac{\cos((2j+1)x)}{(2j+1)^4}.$$

Its derivative is $x(\pi - x)/2$, zero at both ends, and $-u'' = x - \pi/2$. The mean of the first two terms is $\pi^3/24$, canceled by the constant. Twice integrating its coefficient integral by parts using $u' = 0$ at the endpoints gives $n^2 a_n = F_n$, verifying the coefficients. The continuous piecewise smooth even extension identifies the series sum.

Step 4: Separate the free constant from compatibility. A difference satisfies $w'' = 0$ and zero endpoint derivatives, hence is constant. Its zero mean forces it to vanish. Without normalization, all solutions are the displayed polynomial plus any real constant. Adding a constant changes neither the equation nor its derivative conditions, so it cannot fix the integrated forcing balance.