

Question 1

For $f(x) = x$ on $[0, \pi]$, use the convention

$$C_N = \frac{a_0}{2} + \sum_{n=1}^N a_n \cos nx, \quad a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos nx \, dx \quad (n \geq 0).$$

For piecewise smooth periodic extensions, the Fourier series converges to the average of its one-sided limits.

Tasks

1. Compute a_0 and all a_n , and identify the actual constant term. Explain the error caused by confusing a_0 with the mean.
2. Describe the even 2π -periodic extension and determine the sum at the endpoints and inside the interval.
3. Prove uniform convergence on $[0, \pi]$ and give an explicit uniform tail bound. Sketch C_{12} and f .
4. Every C_N has zero endpoint derivatives. Does the limit inherit these slopes? Give a quantitative obstruction to uniform convergence of the derivatives to $f' = 1$.

Question 1 – Solution

Strategy. Separate the mean convention from positive-frequency normalization, and distinguish function convergence from derivative convergence.

Step 1: Compute the mean and coefficients. The zero-frequency integral gives $a_0 = \pi$, so the mean term is $\pi/2$. Integration by parts for $n \geq 1$ gives

$$\int_0^\pi x \cos nx \, dx = \frac{(-1)^n - 1}{n^2}, \quad \boxed{a_n = \frac{2((-1)^n - 1)}{\pi n^2}}.$$

Even modes vanish; odd coefficients are $-4/(\pi n^2)$. Using a_0 in place of $a_0/2$ adds the erroneous constant $\pi/2$.

Step 2: Identify the extension and its limits. The extension is $|x|$ on $[-\pi, \pi]$, repeated with period 2π . It is continuous, with corners at multiples of π . At zero its two limits are zero; at π both are π . Therefore

$$\boxed{x = \frac{\pi}{2} - \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{\cos((2j+1)x)}{(2j+1)^2}, \quad 0 \leq x \leq \pi.}$$

Step 3: Bound the uniform error. The bound $|a_n| \leq 4/(\pi n^2)$ gives absolute uniform convergence by the Weierstrass test. The Fourier convergence theorem identifies the sum with f , including the endpoints. For $N \geq 1$,

$$\boxed{\|f - C_N\|_\infty \leq \frac{4}{\pi} \sum_{n>N} n^{-2} \leq \frac{4}{\pi N}}.$$

Step 4: Reject an unjustified derivative limit. Every finite derivative is a sine sum, so $C'_N(0) = C'_N(\pi) = 0$. The target has one-sided endpoint derivatives one. Thus $\boxed{\sup_{[0,\pi]} |C'_N - 1| \geq 1}$ for every N . Uniform convergence of functions alone does not permit differentiating the limit or passing endpoint derivative conditions to it.

