

Question 9

For $0 < \varepsilon < \pi/2$, define a unit-area narrow pulse centered at $\pi/2$:

$$f_\varepsilon(x) = \begin{cases} (2\varepsilon)^{-1}, & |x - \pi/2| < \varepsilon, \\ 0, & \text{otherwise,} \end{cases} \quad 0 \leq x \leq \pi.$$

Use $b_n(\varepsilon) = (2/\pi) \int_0^\pi f_\varepsilon \sin nx \, dx$. You may use Parseval and Bessel's inequality for square-integrable functions.

Tasks

1. Verify the area and derive all sine coefficients. Interpret the value of $\sin z/z$ at $z = 0$ by its limit.
2. Find the limit of each fixed coefficient as $\varepsilon \downarrow 0$. Prove that the limiting coefficient sequence cannot be the sine coefficients of any $L^2(0, \pi)$ function.
3. Compute $\|f_\varepsilon\|_2^2$ and use Parseval to find $\sum_{n \geq 1} b_n(\varepsilon)^2$. Explain why a point-mass limit is not an ordinary square-integrable function.
4. Use $|\sin z/z - 1| \leq z^2/6$ to bound the largest coefficient error among $1 \leq n \leq N$. Derive this inequality from $\sin z/z = \int_0^1 \cos(zt) \, dt$ and explain why the resulting estimate is not uniform over all mode numbers.

Question 9 – Solution

Strategy. Separate a limit at each fixed frequency from a limit of the complete coefficient sequence in squared norm.

Step 1: Integrate the narrow pulse. The height times the width is one. Writing $a = \pi/2$, integration gives

$$b_n(\varepsilon) = \frac{\cos(n(a - \varepsilon)) - \cos(n(a + \varepsilon))}{\pi\varepsilon n} = \boxed{\frac{2}{\pi} \sin(n\pi/2) \frac{\sin(n\varepsilon)}{n\varepsilon}}.$$

The factor $\sin z/z$ has limiting value one at zero. Even modes vanish.

Step 2: Examine the fixed-mode limit. For each fixed n , $b_n(\varepsilon) \rightarrow \beta_n = (2/\pi) \sin(n\pi/2)$. Every odd-indexed β_n has magnitude $2/\pi$, so $\sum \beta_n^2$ diverges. Bessel's inequality would bound this sum for any L^2 function. Hence no square-integrable function has all these limiting coefficients.

Step 3: Track the diverging energy. Direct integration gives $\boxed{\|f_\varepsilon\|_2^2 = 1/(2\varepsilon)}$. Parseval therefore implies

$$\boxed{\sum_{n \geq 1} b_n(\varepsilon)^2 = \frac{1}{\pi\varepsilon}}.$$

The energies grow without bound while area stays one. Indeed, integration against a continuous test function tends to its value at $\pi/2$, by continuity and averaging over the shrinking interval. This is the point-mass limit, not an L^2 function or an ordinary pointwise sine representation of one.

Step 4: Quantify finite-band convergence. From $|1 - \cos s| \leq s^2/2$ and the supplied integral representation, $|\sin z/z - 1| \leq \int_0^1 z^2 t^2/2 dt = z^2/6$. Thus

$$\boxed{\max_{1 \leq n \leq N} |b_n(\varepsilon) - \beta_n| \leq \frac{N^2 \varepsilon^2}{3\pi}}.$$

For each fixed N this tends to zero. It grows with N , so it does not justify uniform convergence across all frequencies or interchange with the infinite energy sum.