

Question 7

On $[0, 1]$, approximate $f(x) = e^x$ using sine functions $\sin(n\pi x)$. To preserve the endpoint data, also consider

$$\ell(x) = 1 + (e - 1)x, \quad r(x) = e^x - \ell(x), \quad A_N(x) = \ell(x) + \sum_{n=1}^N c_n \sin(n\pi x),$$

where c_n are the sine coefficients of r .

Tasks

1. Compute the sine coefficients b_n of e^x exactly. State the ordinary sine-series limits at zero and one.
2. Compute the sine coefficients of ℓ and subtract them to derive c_n . Explain the improvement in decay.
3. Prove that A_N converges uniformly to e^x on $[0, 1]$, with error at most $(1 + e)/(\pi^3 N^2)$. Find an integer N certified to give error below 10^{-3} .
4. Compare endpoint errors of the raw sine partial sum and A_N . Sketch their errors for $N = 12$, and explain why adding the boundary interpolant changes the approximation problem rather than contradicting the endpoint limitation of pure sine sums.

Question 7 – Solution

Strategy. Remove the nonzero boundary values before asking a sine series for uniform approximation.

Step 1: Compute the raw coefficients. For $k = n\pi$, an antiderivative is $e^x(\sin kx - k \cos kx)/(1 + k^2)$. Evaluation gives

$$b_n = \frac{2k(1 - (-1)^n e)}{1 + k^2}.$$

The odd periodic extension has opposite one-sided values at each endpoint, so its jump averages are zero. Every raw sine partial sum is also zero there.

Step 2: Cancel the leading boundary contribution. Integration of the linear interpolant gives $b_n^\ell = 2(1 - (-1)^n e)/k$. Therefore

$$c_n = b_n - b_n^\ell = -\frac{2(1 - (-1)^n e)}{k(1 + k^2)}.$$

The raw coefficients are of order $1/n$, while c_n are of order $1/n^3$. The residual has zero values at both endpoints; the leading boundary term cancels.

Step 3: Establish a uniform certificate. Since $|c_n| \leq 2(1 + e)/(\pi^3 n^3)$, the residual series converges absolutely and uniformly. Its continuous piecewise smooth odd extension identifies the sum with r throughout the closed interval. Thus

$$\|e^x - A_N\|_\infty \leq \frac{1 + e}{\pi^3 N^2}.$$

For $N = 12$, the bound is below 10^{-3} , so twelve modes suffice.

Step 4: Interpret the endpoint comparison. For the raw sum S_N , $f - S_N$ equals one at zero and e at one. For A_N , both endpoint errors are exactly zero, and the entire error has the small bound above. The graph shows these two errors on the same scale. The corrected approximant includes the non-sine function ℓ ; it is not a pure sine sum and therefore is not constrained to vanish at the endpoints.

