

Question 6

For $0 < x < \pi$, the sine series of $f(x) = x$ is

$$x = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx).$$

Its partial sums converge to f in $L^2(0, \pi)$ as well as pointwise inside. Here $\|g\|_2 = (\int_0^\pi |g|^2 dx)^{1/2}$.

Tasks

1. Form the series obtained by formally differentiating once. Test the necessary term-to-zero condition at $x = \pi/2$ and decide whether the formal differentiation is valid there.
2. Integrate each finite partial sum from zero to x . Use Cauchy–Schwarz and the given L^2 convergence to justify passing to the limit uniformly for $0 \leq x \leq \pi$.
3. Derive the resulting cosine identity for $x^2/2$, including both endpoints. Give a uniform tail bound for this integrated series.
4. Evaluate the identity at $x = \pi$ to sum the reciprocal squares of the positive odd integers. Explain why integration is justified here while the attempted differentiation fails.

Question 6 – Solution

Strategy. Test the operation itself: differentiating weakens coefficient decay, while integration improves it.

Step 1: Reject the formal derivative at a specific point. Formal differentiation produces

$$2 \sum_{n \geq 1} (-1)^{n+1} \cos(nx).$$

At $x = \pi/2$, the even-indexed terms have magnitude two and do not tend to zero. Thus this series diverges there and cannot be a valid ordinary termwise derivative representation of $f' = 1$.

Step 2: Pass from finite integrals to the limit. For each finite sum S_N , direct integration gives

$$\int_0^x S_N(t) dt = 2 \sum_{n=1}^N \frac{(-1)^{n+1}}{n^2} (1 - \cos nx).$$

Cauchy-Schwarz yields $|\int_0^x (S_N - f) dt| \leq \sqrt{x} \|S_N - f\|_2 \leq \sqrt{\pi} \|S_N - f\|_2 \rightarrow 0$. This bound is independent of x , proving uniform convergence of the integrals.

Step 3: State the integrated identity and its error. Consequently

$$\boxed{\frac{x^2}{2} = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} (1 - \cos nx), \quad 0 \leq x \leq \pi.}$$

The tail is bounded uniformly by $4 \sum_{n > N} n^{-2} \leq 4/N$. This also proves absolute uniform convergence of the integrated series. At zero both sides vanish; the right endpoint is included by the uniform integral argument, despite the original sine mismatch.

Step 4: Evaluate at the right endpoint. At $x = \pi$ the even terms vanish and each odd term contributes $4/n^2$. Hence $\pi^2/2 = 4 \sum_{j \geq 0} (2j+1)^{-2}$, giving $\boxed{\sum_{j \geq 0} (2j+1)^{-2} = \pi^2/8}$. The integration has a proved limiting argument and summable coefficients; the differentiated series fails even a necessary condition for convergence.