

Question 5

Let $f(x) = x(\pi - x)$ on $[0, \pi]$ and $S_N = \sum_{n=1}^N b_n \sin(nx)$. You may use Parseval's identity $\int_0^\pi f^2 dx = (\pi/2) \sum_{n \geq 1} b_n^2$, and the standard convergence theorem for piecewise smooth odd periodic extensions.

Tasks

1. Derive all coefficients by integration by parts, and identify the missing modes.
2. Prove a uniform tail bound $\|f - S_N\|_\infty \leq 4/(\pi N^2)$ for $N \geq 1$. State why uniform convergence includes both endpoints here.
3. Find the smallest integer N certified by that bound to give error below 10^{-3} . Distinguish a sufficient certificate from the actual smallest successful truncation.
4. Evaluate $\int_0^\pi f^2 dx$ exactly and use Parseval to derive the sum of the reciprocal sixth powers of the positive odd integers.

Question 5 – Solution

Strategy. Use the zero endpoint values and the constant second derivative to obtain rapidly decaying coefficients.

Step 1: Integrate twice and retain the parity. Because $f(0) = f(\pi) = 0$ and $f'' = -2$, integration by parts gives $\int_0^\pi f \sin(nx) dx = 2(1 - (-1)^n)/n^3$. Therefore

$$b_n = \frac{4(1 - (-1)^n)}{\pi n^3} = \begin{cases} 8/(\pi n^3), & n \text{ odd,} \\ 0, & n \text{ even.} \end{cases}$$

Step 2: Bound the entire tail uniformly. The coefficient series is absolutely summable, so it converges uniformly. The continuous odd periodic extension and the convergence theorem identify its sum with f everywhere on $[0, \pi]$. For every x ,

$$|f(x) - S_N(x)| \leq \frac{8}{\pi} \sum_{n>N} \frac{1}{n^3} \leq \frac{8}{\pi} \int_N^\infty t^{-3} dt = \boxed{\frac{4}{\pi N^2}}.$$

Both endpoint values of the target and all partial sums are zero; there is no endpoint mismatch or jump obstructing uniform convergence.

Step 3: Choose a certified truncation. The stated bound is below 10^{-3} when $N > \sqrt{4000/\pi}$. Since this threshold lies between 35 and 36, $\boxed{N = 36}$ is the smallest integer certified by this bound. The estimate discarded the missing even modes and possible cancellation; it does not prove that every smaller N fails.

Step 4: Extract an exact numerical sum. Expanding the polynomial square yields $\int_0^\pi x^2(\pi - x)^2 dx = \pi^5/30$. Parseval gives

$$\frac{\pi^5}{30} = \frac{32}{\pi} \sum_{j=0}^{\infty} \frac{1}{(2j+1)^6}, \quad \boxed{\sum_{j=0}^{\infty} \frac{1}{(2j+1)^6} = \frac{\pi^6}{960}}.$$

This conclusion uses the complete series and Parseval, not a finite numerical fit.