

Question 4

On $[0, \pi]$, let f equal one on $(\pi/3, 2\pi/3)$ and zero elsewhere, including the two jump points. Let S_N denote its first N sine modes.

Tasks

1. Derive every sine coefficient from the defining integral. Use symmetry about $\pi/2$ to explain why all even modes vanish.
2. Determine whether any odd modes vanish. State the sine-series limit at every point of $[0, \pi]$, paying attention to the assigned values at the jumps.
3. Change the assigned values at the two jumps to arbitrary real numbers. Explain what happens to the coefficients, the series and its pointwise limits.
4. Prove that no continuous function can approximate this assigned target with uniform error less than $1/2$. Explain the implication for sine partial sums and sketch S_{15} alongside the pulse.

Question 4 – Solution

Strategy. Use symmetry for coefficient selection, but use one-sided limits to determine convergence at jumps.

Step 1: Compute and simplify the coefficients. Integration over the nonzero portion gives

$$b_n = \frac{2}{\pi n} [\cos(n\pi/3) - \cos(2n\pi/3)].$$

The reflection $f(\pi - x) = f(x)$ implies $b_n = (-1)^{n+1}b_n$, so even modes are zero. For odd n , $\cos(2n\pi/3) = -\cos(n\pi/3)$, giving $b_n = 4\cos(n\pi/3)/(\pi n)$.

Step 2: Account for every mode and every limit. For odd integers n , $\cos(n\pi/3)$ is $1/2$ or -1 , never zero. Thus no odd mode vanishes. The odd periodic extension is piecewise constant; the Fourier convergence theorem gives one inside the pulse, zero outside, and $1/2$ at $\pi/3$ and $2\pi/3$. Both endpoints have limit zero. The jump-point sums differ from the assigned target values zero.

Step 3: Change only the assigned values. Changing finitely many points does not change any coefficient integral. The partial sums and their limits are therefore unchanged. The series records the jump averages, not arbitrary isolated assignments at the jumps.

Step 4: Establish the uniform-error obstruction. Let g be continuous and suppose $\sup |f - g| \leq E$. Taking one-sided limits at the first jump gives $|g(\pi/3)| \leq E$ and $|1 - g(\pi/3)| \leq E$. The triangle inequality yields $1 \leq 2E$, so $E \geq 1/2$. Every finite sine sum is continuous and obeys this lower bound. Thus pointwise convergence away from the jumps cannot become uniform convergence to this target on the whole interval. The graph shows the finite-sum transition and oscillations.

