

Question 3

A tent on $[0, 1]$ has height $H > 0$ and peak at $a \in (0, 1)$:

$$f(x) = \begin{cases} Hx/a, & 0 \leq x \leq a, \\ H(1-x)/(1-a), & a \leq x \leq 1. \end{cases}$$

Its sine coefficients are $b_n = 2 \int_0^1 f(x) \sin(n\pi x) dx$.

Tasks

1. Integrate separately on the two sides of the corner to derive b_n . Explain why the corner contributes even though both endpoint values vanish.
2. For $a = 1/3, H = 1$, identify all missing modes and prove that the sine series converges uniformly to the tent on the entire closed interval.
3. Suppose the only known coefficient data are $b_1 > 0$ and $b_2/b_1 = 1/4$, and the signal is known to be a tent of this form. Recover a, H uniquely in terms of b_1 .
4. Compare a tent with its reflection $f(1-x)$. State how each coefficient changes and explain why the absolute values of all coefficients cannot distinguish the two peak locations. Sketch the $a = 1/3, H = 1$ tent and its first twelve-mode sum.

Question 3 – Solution

Strategy. Retain the change in slope at the interior corner when integrating by parts.

Step 1: Compute the corner contribution. Put $k = n\pi$. One integration by parts removes the zero endpoint terms; integrating each constant slope then gives

$$\int_0^1 f \sin(kx) dx = \frac{1}{k^2} \left(\frac{H}{a} + \frac{H}{1-a} \right) \sin(ka).$$

Thus $b_n = 2H \sin(n\pi a) / [\pi^2 a(1-a)n^2]$. The nonzero slope change, not an endpoint displacement, supplies these terms.

Step 2: Identify missing modes and convergence. For $a = 1/3, H = 1$, $b_n = 9 \sin(n\pi/3) / (\pi^2 n^2)$. Exactly the multiples of three vanish. The bound $|b_n| \leq 9/(\pi^2 n^2)$ gives uniform absolute convergence by the Weierstrass test. The continuous, piecewise linear odd extension satisfies the Fourier convergence theorem, so the sum is the tent at every point, including both endpoints and the corner.

Step 3: Recover the unknown tent. Because $\sin(\pi a) > 0$, $b_2/b_1 = \sin(2\pi a)/(4 \sin \pi a) = \frac{1}{2} \cos \pi a$. The measured ratio gives $\cos \pi a = 1/2$, hence $a = 1/3$ uniquely on $(0, 1)$. Substituting into b_1 gives

$$H = \frac{2\pi^2 b_1}{9\sqrt{3}}.$$

Step 4: Interpret reflection and lost sign information. Under $x \mapsto 1-x$, $\sin(n\pi(1-x)) = (-1)^{n+1} \sin(n\pi x)$. A change of variable gives $b_n^{\text{reflected}} = (-1)^{n+1} b_n$. Odd modes are unchanged and even modes reverse sign. All magnitudes therefore coincide, so they cannot distinguish a from $1-a$ when those locations differ.

