

Question 2

Let $f(x) = 1$ for $0 < x < \pi$, with assigned endpoint values zero. The odd 2π -periodic extension is a square wave. Define its first M nonzero modes by

$$P_M(x) = \frac{4}{\pi} \sum_{j=0}^{M-1} \frac{\sin((2j+1)x)}{2j+1}.$$

You may use convergence to the average of the one-sided limits for piecewise smooth periodic functions.

Tasks

1. Derive the sine coefficients and verify the definition of P_M . State the limiting values inside the interval and at its endpoints.
2. Sum the finite cosine progression to prove $P'_M(x) = (2/\pi) \sin(2Mx)/\sin x$ for $0 < x < \pi$. Locate the first local maximum to the right of zero.
3. Express the height of that maximum as an integral. Rescale the integral and prove that the heights tend to $(2/\pi) \int_0^\pi (\sin t)/t \, dt$.
4. Prove that this limiting height is greater than one, using $\sin t/t \geq 1 - t^2/6$ for $0 \leq t \leq \pi/2$ and $1/t \geq 1/\pi$ for $\pi/2 \leq t \leq \pi$. Sketch P_{10} and explain why pointwise convergence does not remove the near-jump overshoot.

Question 2 – Solution

Strategy. Analyze a moving maximum of a finite sum rather than substitute a moving point into a pointwise convergence claim.

Step 1: Find the nonzero modes. Direct integration gives $b_n = 2(1 - (-1)^n)/(\pi n)$: zero for even n and $4/(\pi n)$ for odd n . Thus P_M is the stated truncation. The odd extension has jump averages zero at both endpoints; its sum is one on $(0, \pi)$.

Step 2: Locate the first maximum. A finite geometric sum, or telescoping $2 \sin x \cos((2j+1)x)$, gives $\sum_{j=0}^{M-1} \cos((2j+1)x) = \sin(2Mx)/(2 \sin x)$. Differentiating the finite sum is legitimate, so

$$P'_M(x) = \frac{2}{\pi} \frac{\sin(2Mx)}{\sin x}, \quad x_M = \frac{\pi}{2M}.$$

The derivative is positive before x_M and negative just after it, for $M \geq 1$.

Step 3: Pass to the moving-peak limit correctly. Since $P_M(0) = 0$, substitution $t = 2Mx$ gives

$$P_M(x_M) = \frac{2}{\pi} \int_0^\pi \frac{\sin t}{2M \sin(t/(2M))} dt \longrightarrow \boxed{\frac{2}{\pi} \int_0^\pi \frac{\sin t}{t} dt}.$$

Write the integrand as $(\sin t/t)(z/\sin z)$, where $z = t/(2M)$. Since $z/\sin z$ extends continuously with value one at zero, this factor tends uniformly to one for $0 \leq t \leq \pi$. Also $|\sin t/t| \leq 1$. These facts justify the integral limit; its value is approximately 1.179.

Step 4: Prove a persistent excess. The supplied inequalities imply $\int_0^\pi \sin t/t dt \geq \pi/2 - \pi^3/144 + 1/\pi > \pi/2$, since $\pi^4 < 144$. The limiting peak is strictly above one. The peak location moves toward the jump, so convergence at each fixed interior point is consistent with this persistent overshoot. The square wave jumps by two across zero.

