

Question 1

Let $f(x) = x$ on $[0, \pi]$. Define its sine coefficients and partial sums by

$$b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin(nx) dx, \quad S_N = \sum_{n=1}^N b_n \sin(nx).$$

For a piecewise smooth odd periodic extension, the Fourier series converges to the average of its one-sided limits, including at periodic joins.

Tasks

1. Derive b_n by integration by parts. Describe the odd 2π -periodic extension, including its jump at odd multiples of π .
2. Determine the series sum at every $x \in [0, \pi]$. Distinguish the assigned endpoint value $f(\pi)$ from the series limit.
3. Decide whether S_N converges uniformly to f on $[0, \pi]$, or to x on the open interval $(0, \pi)$. Give a quantitative obstruction for both claims.
4. Use the value at $x = \pi/2$ to evaluate $1 - 1/3 + 1/5 - \dots$. Sketch S_{12} with f and explain why the endpoint mismatch does not invalidate the interior representation.

Question 1 – Solution

Strategy. Determine the odd extension before applying the convergence theorem at its endpoints.

Step 1: Compute the coefficients and extension. Integration by parts yields

$$\int_0^{\pi} x \sin(nx) dx = [-x \cos(nx)/n]_0^{\pi} + [\sin(nx)/n^2]_0^{\pi} = -\pi(-1)^n/n.$$

Hence $b_n = 2(-1)^{n+1}/n$. The odd extension equals x on $(-\pi, \pi)$ and repeats every 2π . At an odd multiple of π its left and right limits are π and $-\pi$. The join value may be chosen zero for an odd periodic representative.

Step 2: Apply the convergence theorem. The series equals x for $0 < x < \pi$. At zero it equals zero by continuity of the extension, and at π it equals the jump average zero:

$$\sum_{n \geq 1} \frac{2(-1)^{n+1}}{n} \sin(nx) = x \quad (0 \leq x < \pi), \quad \text{sum at } \pi = 0.$$

This does not reproduce the separately assigned value $f(\pi) = \pi$.

Step 3: Rule out uniform convergence near the join. On the closed interval, $|f(\pi) - S_N(\pi)| = \pi$ for every N . On the open interval, continuity of each finite sum gives $\lim_{x \uparrow \pi} |x - S_N(x)| = \pi$, so the supremum error is at least π . Neither asserted uniform convergence holds, despite pointwise convergence inside.

Step 4: Evaluate an interior numerical series. At $x = \pi/2$, only odd modes remain and their sine values alternate: $\pi/2 = 2(1 - 1/3 + 1/5 - \dots)$. Therefore the convergent series equals $\pi/4$. The graph displays a finite approximation; the convergence theorem, not endpoint agreement of the original data, justifies the interior sum.

