

Question 10

Let $u(x) = \sin x$ and $v(x) = \sin 5x$ on $[0, 2\pi]$. Compare the continuous average inner product with its equally spaced sampling version:

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} fg \, dx, \quad \langle f, g \rangle_N = \frac{1}{N} \sum_{j=0}^{N-1} f(2\pi j/N)g(2\pi j/N).$$

The sampled form may fail to be positive definite on functions even though it is positive definite on the vectors of sampled values.

Tasks

1. Compute the continuous Gram matrix of u, v ; a Gram matrix has entries consisting of all pairwise inner products.
2. For $N = 4$, list both sample vectors and compute the sampled Gram matrix. Exhibit a nonzero continuous function in their span with zero sampled norm, and sketch the curves with the common sampled points.
3. For integer m , derive $S_m = N^{-1} \sum_{j=0}^{N-1} \cos(2\pi m j/N)$ using a finite geometric sum. Express every entry of the sampled Gram matrix in terms of S_2, S_4, S_6, S_{10} .
4. Find the smallest integer $N \geq 3$ for which the sampled Gram matrix equals the continuous one. Check every smaller candidate and explain why exactness for this pair does not imply exactness for all continuous functions.

Question 10 – Solution

Strategy. Distinguish continuous orthogonality from orthogonality of a finite set of sampled values.

Step 1: Compute the continuous matrix. Product-to-sum gives $\langle u, v \rangle = 0$, while each sine has average square $1/2$. Hence $G = \frac{1}{2}I_2$.

Step 2: Identify the four-point alias. At $0, \pi/2, \pi, 3\pi/2$, both functions give $(0, 1, 0, -1)$. Thus

$$G_4 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

The nonzero function $u - v$ vanishes at every sample, so its sampled norm is zero, although its continuous squared norm is one. The samples have lost the distinction.

Step 3: Derive the exact sampling formulas. Put $z = e^{2\pi im/N}$. If N divides m , every summand equals one. Otherwise $z \neq 1$ and $\sum_{j=0}^{N-1} z^j = (1 - z^N)/(1 - z) = 0$. Taking real parts yields $S_m = 1$ when $N \mid m$, and $S_m = 0$ otherwise. Product-to-sum now gives

$$G_N = \frac{1}{2} \begin{pmatrix} 1 - S_2 & S_4 - S_6 \\ S_4 - S_6 & 1 - S_{10} \end{pmatrix}.$$

Step 4: Find the smallest exact sample count. For $N = 3$ the off-diagonal entry is $-1/2$; for $N = 4$ it is $1/2$. For $N = 5$, the second diagonal entry is zero. For $N = 6$ the off-diagonal entry is $-1/2$. None matches G . For $N = 7$, none of $2, 4, 6, 10$ is divisible by seven, so $G_7 = G$ and seven is the smallest allowed count. This is exact only for the tested products: for example, $\sin 7x$ vanishes at all seven sample points but has continuous average square $1/2$.

