

Question 8

Approximate $f(x) = |x|$ on $[-1, 1]$ by a polynomial $g(x) = a + bx^2$. Use the squared error $\|f - g\|_2^2 = \int_{-1}^1 (f - g)^2 dx$. For this inner product, 1 and $p_2 = x^2 - 1/3$ are orthogonal, with squared norms 2 and $8/45$ respectively.

Tasks

1. Compute the projection coefficients onto 1, p_2 and find the unique polynomial g_* minimizing the squared integral error.
2. Verify directly that the residual is orthogonal to both 1 and x^2 . Prove the error decomposition that establishes the minimum and uniqueness.
3. Compute the exact minimum squared error. Sketch f and g_* , and identify where the largest absolute residual occurs.
4. Does the least-squares optimum also minimize the maximum absolute error? Compare it with $h(x) = 1/8 + x^2$ and justify your conclusion by exact uniform error bounds.

Question 8 – Solution

Strategy. Use an orthogonal basis for the integral objective, then test the distinct uniform-error objective separately.

Step 1: Compute the projection. The needed moments are $\langle f, 1 \rangle = 1$ and $\langle f, p_2 \rangle = 1/2 - 1/3 = 1/6$. Thus

$$g_* = \frac{1}{2} + \frac{1/6}{8/45}(x^2 - 1/3) = \boxed{\frac{3}{16} + \frac{15}{16}x^2}.$$

Step 2: Verify residual orthogonality and uniqueness. For $r = f - g_*$,

$$\int_{-1}^1 r \, dx = 1 - \frac{3}{8} - \frac{5}{8} = 0, \quad \int_{-1}^1 rx^2 \, dx = \frac{1}{2} - \frac{1}{8} - \frac{3}{8} = 0.$$

For any other $g = a + bx^2$, the cross term vanishes, so $\|f - g\|_2^2 = \|r\|_2^2 + \|g_* - g\|_2^2$. The second term is zero only when the two continuous polynomials agree, proving both optimality and uniqueness.

Step 3: Evaluate the two errors for the optimum. Subtracting the squared projection norm from $\|f\|_2^2 = 2/3$ gives

$$\|r\|_2^2 = \frac{2}{3} - \frac{1}{2} - \frac{(1/6)^2}{8/45} = \frac{1}{96}.$$

On $[0, 1]$, $r = x - 3/16 - 15x^2/16$ has maximum $19/240$ at $x = 8/15$, and endpoint values $-3/16, -1/8$. By evenness, the largest absolute error is $\|r\|_\infty = 3/16$, attained at zero.

Step 4: Disprove uniform optimality. For $h = 1/8 + x^2$, put $t = |x| \in [0, 1]$. Then $f - h = t(1-t) - 1/8 \in [-1/8, 1/8]$, with both extremes attained. Hence $\|f - h\|_\infty = 1/8 < 3/16$. This competitor proves that the unique least-squares solution is not a uniform best approximation. Different error objectives need not select the same function.

