

Question 7

Let $p \in C^1[a, b]$ be positive, let q, w be continuous with $w > 0$, and let u, v be nonzero real C^2 eigenfunctions:

$$-(pu')' + qu = \lambda wu, \quad -(pv')' + qv = \mu wv.$$

At each endpoint $c = a, b$, both satisfy the same condition $A_c y(c) + B_c p(c) y'(c) = 0$, with real $(A_c, B_c) \neq (0, 0)$.

Tasks

1. Derive an identity relating $(\lambda - \mu) \int_a^b wuv \, dx$ to endpoint terms by multiplying and subtracting the differential equations.
2. Prove the endpoint terms vanish for all the stated boundary conditions, including $B_c = 0$. Deduce weighted orthogonality when $\lambda \neq \mu$.
3. Apply the result to $u = \sin \pi x$ and $v = \sin 2\pi x$ on $[0, 1]$ with zero endpoint values. Give an orthonormal pair.
4. Explain why equal eigenvalues do not force orthogonality. On $[0, 2\pi]$ with periodic value-and-slope matching, use $u = \sin x$ and $v = \sin x + \cos x$ as a counterexample, then orthogonalize and normalize this pair.

Question 7 – Solution

Strategy. Track the boundary expression before dividing by an eigenvalue difference.

Step 1: Derive the boundary identity. Multiply the equation for u by v and that for v by u , then subtract. The q terms cancel, and integration gives

$$(\lambda - \mu) \int_a^b wuv \, dx = [p(uv' - u'v)]_a^b.$$

This identity is valid before any orthogonality conclusion is drawn.

Step 2: Use the actual endpoint conditions. At an endpoint with $B_c = 0$, $A_c \neq 0$ and both function values are zero, so the boundary expression is zero. If $B_c \neq 0$, then $pu' = -A_c u/B_c$ and $p'v = -A_c v/B_c$;

substitution again gives zero. Therefore $\lambda \neq \mu$ implies $\int_a^b wuv \, dx = 0$. Equal eigenvalues give only the identity $0 = 0$.

Step 3: Normalize two distinct Dirichlet modes. Here $p = w = 1, q = 0$, with eigenvalues π^2 and $4\pi^2$. The theorem gives orthogonality, and direct integration of each square gives $1/2$. Thus

$\sqrt{2} \sin \pi x, \sqrt{2} \sin 2\pi x$ are orthonormal. The normalization is separate from the boundary proof of orthogonality.

Step 4: Handle a repeated eigenvalue. Both $u = \sin x$ and $v = \sin x + \cos x$ have eigenvalue one and satisfy periodic matching on $[0, 2\pi]$. Nevertheless, $\langle u, v \rangle = \pi \neq 0$. Since $\|u\|^2 = \pi$, subtracting the projection gives $v - \langle v, u \rangle u / \|u\|^2 = \cos x$. The resulting orthonormal pair is

$$\sin x / \sqrt{\pi}, \quad \cos x / \sqrt{\pi}.$$

Distinct eigenvalues ensure orthogonality under the hypotheses; a repeated eigenspace may instead require an orthogonal choice of basis.