

Question 6

On continuous real functions on $[0, 1]$, define

$$\langle f, g \rangle_w = \int_0^1 x f(x) g(x) dx.$$

The weight vanishes at one endpoint. Seek a monic quadratic $q(x) = x^2 + ax + b$ orthogonal to both 1 and x in this inner product.

Tasks

1. Prove positive definiteness despite the zero weight at $x = 0$. Explain the role of continuity of the functions.
2. Determine a, b from the two weighted moment equations. Verify both equations and prove uniqueness of the monic quadratic.
3. Compute $\|q\|_w^2$, give its unit-norm version with positive leading coefficient, and locate its roots. Sketch q on $[0, 1]$.
4. Test whether the same q is orthogonal to 1 for unweighted integration. Also explain why replacing the weight by $x - 1/2$ would fail to define an inner product on this space.

Question 6 – Solution

Strategy. Include the weight in every moment, and distinguish an isolated zero of a nonnegative weight from a sign-changing weight.

Step 1: Establish positive definiteness. Symmetry and bilinearity follow from integration, and $\int_0^1 x f^2 dx \geq 0$. If a continuous f is not identically zero, continuity supplies an interval of positive length inside $(0, 1]$ on which $f^2 > 0$, even if the first known nonzero value is at zero. The weighted integral is then positive. Thus zero weighted norm forces $f = 0$ everywhere.

Step 2: Solve the weighted moment equations. The two conditions are

$$\frac{1}{4} + \frac{a}{3} + \frac{b}{2} = 0, \quad \frac{1}{5} + \frac{a}{4} + \frac{b}{3} = 0.$$

Their coefficient determinant is $1/9 - 1/8 = -1/72 \neq 0$, so the solution is unique:

$$a = -6/5, \quad b = 3/10, \quad q = x^2 - 6x/5 + 3/10.$$

Substitution gives $1/4 - 2/5 + 3/20 = 0$ and $1/5 - 3/10 + 1/10 = 0$.

Step 3: Normalize and locate the sign changes. Expansion and integration give

$$\int_0^1 x q^2 dx = \frac{1}{6} - \frac{12}{25} + \frac{51}{100} - \frac{6}{25} + \frac{9}{200} = \boxed{\frac{1}{600}}.$$

Thus the unit-norm version is $\boxed{\sqrt{600}q}$. The roots are $\boxed{(6 - \sqrt{6})/10, (6 + \sqrt{6})/10}$, both strictly inside $(0, 1)$. The quadratic is positive outside these roots and negative between them.

Step 4: Check two changes of weight. Without the weight, $\int_0^1 q dx = 1/3 - 3/5 + 3/10 = \boxed{1/30 \neq 0}$. Orthogonality therefore changes with the weight. For the proposed signed weight, the nonzero constant function one would have squared “norm” $\int_0^1 (x - 1/2) dx = 0$. This violates positive definiteness, so that bilinear form is not an inner product on the stated space.

