

Question 5

Use $\langle f, g \rangle = \int_{-1}^1 f(x)g(x)dx$. Construct monic orthogonal polynomials from $1, x, x^2, x^3$; monic means the leading coefficient is one.

Tasks

1. Explain which inner products vanish by parity. Starting with $p_0 = 1$ and $p_1 = x$, find monic p_2, p_3 so that all four polynomials are mutually orthogonal.
2. Compute their squared norms exactly and turn them into an orthonormal set. Distinguish monic normalization from unit-norm normalization.
3. Find the unique quadratic-or-lower polynomial minimizing $\int_{-1}^1 [x^3 - q(x)]^2 dx$. Give the exact minimum error and justify uniqueness.
4. A proposed shortcut merely divides each monomial by its norm. Give an explicit inner product showing why this does not produce an orthogonal family.

Question 5 – Solution

Strategy. Use parity to eliminate unnecessary projections, then compute the remaining moments exactly.

Step 1: Construct the monic family. An odd integrand integrates to zero on $[-1, 1]$, so every even polynomial is orthogonal to every odd polynomial. For $p_2 = x^2 - c$, the equation $\langle p_2, 1 \rangle = 2/3 - 2c = 0$ gives $c = 1/3$. For $p_3 = x^3 - dx$, $\langle p_3, x \rangle = 2/5 - 2d/3 = 0$ gives $d = 3/5$. Thus

$$p_0 = 1, \quad p_1 = x, \quad p_2 = x^2 - 1/3, \quad p_3 = x^3 - 3x/5.$$

Parity and these two moment equations verify every distinct pair.

Step 2: Compute and apply the norms. Integrating the squares term by term yields

$$\|p_0\|^2 = 2, \quad \|p_1\|^2 = 2/3, \quad \|p_2\|^2 = 8/45, \quad \|p_3\|^2 = 8/175.$$

An orthonormal set is therefore $\left[p_0/\sqrt{2}, \sqrt{3/2}p_1, \sqrt{45/8}p_2, \sqrt{175/8}p_3 \right]$. Monic normalization fixes the leading coefficient; unit normalization fixes the squared integral and generally changes that coefficient.

Step 3: Find the best quadratic approximation. The decomposition $x^3 = 3x/5 + p_3$ has its residual orthogonal to every quadratic. For any other quadratic q , the cross term vanishes, giving

$$\|x^3 - q\|^2 = \|p_3\|^2 + \|3x/5 - q\|^2.$$

Hence $\left[q_* = 3x/5, \quad \min \|x^3 - q\|^2 = 8/175 \right]$. Equality forces the continuous polynomial $q - 3x/5$ to have zero norm and therefore vanish identically, proving uniqueness.

Step 4: Reject normalization without orthogonalization. The normalized versions of 1 and x^2 are $1/\sqrt{2}$ and $\sqrt{5/2}x^2$. Their inner product is $\sqrt{5}/3 \neq 0$. Normalizing lengths alone does not remove projections onto the preceding functions.