

Question 4

Orthogonality depends on the interval used in the inner product. For $L > 0$, let

$$\langle u, v \rangle_{a,L} = \int_a^{a+L} u(x)v(x)dx.$$

Examine $u = \sin x$ and $v = \cos x$.

Tasks

1. Compute $\langle u, v \rangle_{a,L}$ exactly. Characterize all lengths L for which u and v are orthogonal for every starting point a .
2. When $L = \pi/2$, find all starting points that give orthogonality. Explain why the same length can work at one location and fail at another.
3. On $[0, \pi/2]$, compute both squared norms and the inner product of the individually normalized functions. Decide whether normalization creates orthogonality.
4. On a window of length $m\pi$, m a positive integer, give an orthonormal pair obtained by rescaling u, v . Sketch their product on $[0, \pi]$ to explain cancellation without treating the graph as proof.

Question 4 – Solution

Strategy. Compute the product integral before deciding whether a pair is orthogonal or merely normalized.

Step 1: Determine the window dependence. Using $\sin x \cos x = \frac{1}{2} \sin 2x$ gives

$$\langle u, v \rangle_{a,L} = \frac{\cos 2a - \cos(2a + 2L)}{4} = \boxed{\frac{1}{2} \sin L \sin(2a + L)}.$$

It vanishes for all a exactly when $\sin L = 0$. The positive lengths are $\boxed{L = m\pi, m = 1, 2, \dots}$. If $\sin L \neq 0$, one can choose a with $\sin(2a + L) = 1$, proving necessity.

Step 2: Locate the shorter orthogonal windows. For $L = \pi/2$, orthogonality requires $\sin(2a + \pi/2) = 0$. Hence

$$\boxed{a = -\pi/4 + j\pi/2, \quad j \in \mathbb{Z}.$$

The product changes sign; shifting the window can balance its signed areas. At $a = 0$, however, the product is positive inside the interval.

Step 3: Test individual normalization. On $[0, \pi/2]$, $\|u\|^2 = \|v\|^2 = \pi/4$ and $\langle u, v \rangle = 1/2$. The unit-norm functions are $\hat{u} = 2 \sin x / \sqrt{\pi}$ and $\hat{v} = 2 \cos x / \sqrt{\pi}$, with

$$\boxed{\langle \hat{u}, \hat{v} \rangle = 2/\pi \neq 0}.$$

Nonzero scaling cannot turn a nonzero inner product into zero.

Step 4: Normalize on the correct windows. Over any length $m\pi$, the oscillatory term in either squared integral cancels, so both squared norms equal $m\pi/2$. Thus $\boxed{\sqrt{2/(m\pi)} \sin x, \sqrt{2/(m\pi)} \cos x}$ are orthonormal on every such window. The graph shows equal positive and negative product areas on $[0, \pi]$; the integral calculation establishes their exact cancellation.

