

Question 3

Define the two-periodic extension of a ramp by the unique representation

$$x = 2m + r, \quad m \in \mathbb{Z}, \quad -1 \leq r < 1, \quad s(x) = r.$$

The half-open convention specifies the actual function values at the joins. Also let p be the one-periodic extension of $x(1 - x)$ from $[0, 1)$.

Tasks

1. Prove that s has fundamental period two. Find its value and one-sided limits at every odd integer, and sketch it on $[-3, 3]$ with correct open and closed points.
2. Determine whether $s(-x) = -s(x)$ holds everywhere. Change only its join values to produce an odd, two-periodic function, and decide whether that change can make it continuous.
3. Prove that $\int_a^{a+2} s(x) dx = 0$ for every real a . Explain why changing the join values does not affect this statement.
4. Determine the continuity and differentiability of p at integers. Explain why matching function values at the ends of a defining interval is weaker than matching first derivatives.

Question 3 – Solution

Strategy. Separate periodicity, assigned endpoint values, one-sided limits and integral properties.

Step 1: Locate the joins and establish minimality. The representation gives $s(x+2) = s(x)$. Its discontinuities occur precisely at odd integers. Any period must preserve this set, so a positive period is a positive even integer; two is therefore fundamental. At $c = 2m + 1$,

$$s(c) = -1, \quad s(c-) = 1, \quad s(c+) = -1.$$

Each increasing segment has its left endpoint included and its right endpoint excluded. The graph also includes the assigned value $s(3) = -1$.

Step 2: Repair oddness, not continuity. Away from odd integers, $s(-x) = -s(x)$. At an odd integer both $s(x)$ and $s(-x)$ equal -1 , so pointwise oddness fails. Assigning zero at every odd integer produces an odd function and preserves two-periodicity. No assigned join value can reconcile the unequal one-sided limits, so the jumps remain.

Step 3: Integrate over any full period. The integral on $[-1, 1]$ is $\int_{-1}^1 x \, dx = 0$. Reduce a modulo two to $b \in [-1, 1)$. Splitting at one and shifting the remaining piece by two gives

$$\int_a^{a+2} s = \int_b^1 s + \int_1^{b+2} s = \int_b^1 s + \int_{-1}^b s = 0.$$

Changing finitely many join values in any bounded interval leaves these Riemann integrals unchanged. Integral cancellation does not require pointwise oddness.

Step 4: Check a continuous periodic join. For p , both endpoint limits of $x(1-x)$ are zero, matching the assigned value at each integer. Thus p is continuous everywhere. The left derivative at an integer is -1 , while the right derivative is 1 , so it is not differentiable there. Value matching alone does not give derivative matching.

