

Question 2

Consider a signal with incommensurate frequencies:

$$f(x) = \cos x + \cos(\sqrt{2}x).$$

A graph over a long interval may suggest repeated patterns without establishing an exact period. Recall that $\sqrt{2}$ is irrational.

Tasks

1. Show that $f(x) \leq 2$ and determine every point where equality holds. Use this to prove that f has no positive period.
2. For a second proof, apply $D^2 + 2$ to f , where $D = d/dx$. Explain why a period of a smooth function is also a period of this derivative combination, and derive a contradiction.
3. Set $T = 140\pi$. Prove a uniform bound on $|f(x+T) - f(x)|$ using $99^2 - 2 \cdot 70^2 = 1$ and $|\cos(u+v) - \cos u| \leq |v|$.
4. Decide whether this small uniform discrepancy makes T an exact period. Contrast the result with $h(x) = \cos x + \cos(3x/2)$, and determine the fundamental period of h .

Question 2 – Solution

Strategy. Distinguish exact simultaneous phase return from a uniformly small phase mismatch.

Step 1: Use the exact maximum. Each cosine is at most one. Equality $f(x) = 2$ requires $x = 2\pi m$ and $\sqrt{2}x = 2\pi n$ for integers m, n . Irrationality forces $m = n = 0$, so the maximum is attained only at $x = 0$. If $T > 0$ were a period, then $f(T) = f(0) = 2$, contradicting this uniqueness.

Step 2: Isolate the frequencies by differentiation. Direct differentiation gives $(D^2 + 2)f = \cos x$. Differentiating an identity $f(x+T) = f(x)$ twice preserves it, so T would also be a period of $\cos x$. Subtracting this cosine from f shows that T preserves $\cos(\sqrt{2}x)$ too. Evaluating both shifted cosines at zero requires $T = 2\pi m$ and $\sqrt{2}T = 2\pi n$, impossible for positive T .

Step 3: Bound an approximate return. At $T = 140\pi$, the first cosine returns exactly. The second phase shift is $2\pi(70\sqrt{2}) = 2\pi \cdot 99 - \delta$, where

$$\delta = 2\pi(99 - 70\sqrt{2}) = \frac{2\pi}{99 + 70\sqrt{2}}.$$

The stated cosine inequality therefore yields

$$\sup_{x \in \mathbb{R}} |f(x + 140\pi) - f(x)| \leq \frac{2\pi}{99 + 70\sqrt{2}} < 0.032.$$

Step 4: Contrast approximate and exact recurrence. The discrepancy bound is positive and does not make T a period; Step 1 rules out every positive exact period. For h , any period must send its value two at zero to another value two. Thus $T = 2\pi m$ and $3T/2 = 2\pi n$, forcing m even. Conversely 4π preserves both summands. Hence $\boxed{T_h = 4\pi}$, an exact fundamental period.