

Question 1

A positive number T is a period of f if $f(x + T) = f(x)$ for every real x . A fundamental period is the smallest positive period, when one exists. Consider

$$f(x) = \sin 6x + \cos 10x.$$

You may use linear independence of sines and cosines at distinct positive integer frequencies; justify that fact here by integrating products on $[0, 2\pi]$.

Tasks

1. Find a common period of the two summands. Explain why a common period alone does not prove that the sum has that fundamental period.
2. Expand $f(x + T) - f(x)$ and use independence to characterize every positive period of f . Determine the fundamental period exactly.
3. Find the fundamental period of $g(x) = 3f(2x - 1) + 7$. Explain the effects of translation, nonzero amplitude scaling and a constant offset.
4. Decide whether a nonzero constant function and the zero function have fundamental periods. Explain why the definition cannot simply assign either function period zero as a fundamental period.

Question 1 – Solution

Strategy. Prove that a period of the sum must preserve each of its distinct frequency components.

Step 1: Find a candidate period. The summands have periods $\pi/3$ and $\pi/5$, and π is a common period. Thus π is a period of f . In general, cancellation can give a sum a shorter period or even make it constant, so minimality still needs proof.

Step 2: Prove necessity and minimality. Product-to-sum identities show that distinct positive-integer sine/cosine frequencies have zero product integral on $[0, 2\pi]$, while each squared integral equals π . Multiplying a zero linear combination by each component and integrating therefore forces every coefficient to vanish. Expanding the shift of f now gives

$$\cos 6T = 1, \quad \sin 6T = 0, \quad \cos 10T = 1, \quad \sin 10T = 0.$$

Equivalently, $6T = 2\pi m$ and $10T = 2\pi n$ for integers m, n . Then $5m = 3n$, so $m = 3j, n = 5j$ and $T = j\pi$. Conversely every such positive shift works. Hence

Positive periods: $j\pi$ ($j = 1, 2, \dots$), $T_0 = \pi$.

Step 3: Transform the independent variable. Since $2x - 1$ ranges over all real numbers, a shift $T > 0$ preserves g exactly when $2T$ preserves f . Thus $T_g = \pi/2$. Translation, multiplication by a nonzero constant and addition of a constant do not change the set of periods; the argument dilation rescales it.

Step 4: Treat constant functions separately. Every positive number is a period of a constant function, including the zero function. There is no smallest positive number, so neither has a fundamental period. Zero is excluded by the definition and cannot be used to repair this lack of a least positive period.