

Question 9

An inverse design seeks a continuous potential q on $[0, 1]$ for

$$-y'' + q(x)y = \lambda y, \quad y(0) = y(1) = 0.$$

For the constructive part, prescribe $\lambda = \pi^2$ and $y = \sin \pi x + \frac{1}{4} \sin 2\pi x$. Eigenfunctions are required to be C^2 .

Tasks

1. Show that the alternative target $y = x(1 - x)$ cannot be an eigenfunction for any continuous q and any finite real λ . Explain the endpoint obstruction.
2. For the trigonometric target, recover q explicitly and extend it continuously to both endpoints. Compute $q(0)$ and $q(1)$ and verify the differential equation.
3. Prove that the target is positive on $(0, 1)$ and has simple endpoint zeros. For any C^2 test function v with zero endpoints, put $z = v/y$ inside the interval and derive $\int_0^1 [v'^2 + (q - \lambda)v^2]dx = \int_0^1 y^2 z'^2 dx$.
4. Use this identity to prove that no real eigenvalue can lie below π^2 , and that every eigenfunction with eigenvalue π^2 is a multiple of the designed target. Address the boundary term in the identity.

Question 9 – Solution

Strategy. Recover the potential where the target is nonzero, then use endpoint regularity and a factored energy identity to certify the design.

Step 1: Reject an incompatible target. For $y = x(1 - x)$, division inside the interval would require $q = \lambda + y''/y = \lambda - 2/[x(1 - x)]$, which is unbounded at both ends. Equivalently, continuity of the equation at zero would require $-y''(0) + q(0)y(0) = 2 = \lambda y(0) = 0$. No continuous potential can work.

Step 2: Construct a regular potential. Write $c = \cos \pi x$. The target is $y = \sin \pi x(1 + c/2)$ and $y'' = -\pi^2 \sin \pi x(1 + 2c)$. Consequently

$$q(x) = -\frac{3\pi^2 c}{2 + c}, \quad q(0) = -\pi^2, \quad q(1) = 3\pi^2.$$

Since $2 + c \geq 1$, the formula is continuous on the closed interval. Substitution gives $qy = (\lambda y + y'')$, verifying the equation inside; continuity verifies it at the endpoints as well.

Step 3: Factor the energy. The factor $1 + c/2$ lies between $1/2$ and $3/2$, so $y > 0$ inside. The endpoint derivatives are $y'(0) = 3\pi/2$ and $y'(1) = -\pi/2$, both nonzero. With $v = yz$ and $y'' = (q - \lambda)y$, expansion gives

$$v'^2 + (q - \lambda)v^2 = y^2 z'^2 + (yy' z^2)'$$

For zero-endpoint C^2 functions v , the simple zeros of y imply that $z = v/y$ and z' stay bounded near either endpoint, by Taylor expansion. Thus $[yy' z^2]_0^1 = 0$ and integration proves the requested identity.

Step 4: Certify the lowest eigenvalue and its eigenspace. If v is a nonzero eigenfunction with eigenvalue ν , integration by parts and the identity yield

$$(\nu - \pi^2) \int_0^1 v^2 dx = \int_0^1 y^2 z'^2 dx \geq 0.$$

Thus $\nu \geq \pi^2$, and the constructed nonzero y attains this bound. Equality forces $z' = 0$ inside, hence $v = Cy$. Therefore π^2 is the lowest real eigenvalue and its eigenspace is precisely $\text{span}\{y\}$.