

Question 8

Compare an eigenproblem with its forced counterpart on $[0, 1]$:

$$-y'' - \lambda y = \sin(\pi x), \quad y(0) = y(1) = 0.$$

The homogeneous Dirichlet eigenvalues of $-d^2/dx^2$ are $n^2\pi^2$, $n \geq 1$. The identity

$$\int_0^1 \sin(n\pi x) \sin(\pi x) dx = 0 \quad (n \geq 2)$$

may be verified by $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$.

Tasks

1. Solve the forced problem for every real λ outside the homogeneous spectrum and prove uniqueness.
2. At $\lambda = \pi^2$, prove nonexistence using integration by parts against $\sin(\pi x)$, rather than merely observing a vanishing denominator.
3. At $\lambda = n^2\pi^2$ with $n \geq 2$, find all solutions. Determine which is selected by $\int_0^1 y \sin(n\pi x) dx = 0$.
4. Compare the uniform size of the solution as $\lambda \rightarrow \pi^2$ with its limit as $\lambda \rightarrow 4\pi^2$ through nonspectral values. Explain why a finite limiting particular solution does not imply uniqueness at the limiting parameter.

Question 8 – Solution

Strategy. Separate the existence of a particular solution from the nontrivial homogeneous space at a spectral parameter.

Step 1: Solve away from the spectrum. Since $-(\sin \pi x)'' = \pi^2 \sin \pi x$,

$$y = \frac{\sin \pi x}{\pi^2 - \lambda}$$

solves the equation and both endpoints whenever $\lambda \neq \pi^2$. If λ is outside the entire spectrum, any difference of two solutions is a zero-endpoint homogeneous solution and is therefore zero. This proves uniqueness, including all zero and negative parameter values.

Step 2: Prove failure of compatibility at the first eigenvalue. Let $s = \sin \pi x$. For a proposed solution at $\lambda = \pi^2$, twice integrating by parts gives

$$\int_0^1 s(-y'' - \pi^2 y) dx = [-sy' + s'y]_0^1 + \int_0^1 (-s'' - \pi^2 s)y dx = 0.$$

Both s and y vanish at the endpoints. The forcing instead requires this integral to equal $\int_0^1 s^2 dx = 1/2$, a contradiction. No solution exists.

Step 3: Find every compatible resonant solution. For $n \geq 2$, the particular solution above is valid and the homogeneous space is spanned by $\sin(n\pi x)$. Therefore all solutions are

$$y = \frac{\sin \pi x}{(1 - n^2)\pi^2} + C \sin(n\pi x), \quad C \in \mathbb{R}.$$

The supplied product identity and $\int_0^1 \sin^2(n\pi x) dx = 1/2$ show that the extra integral condition is $C/2 = 0$, selecting exactly $C = 0$.

Step 4: Distinguish two limiting behaviors. Outside the spectrum, $\max |y| = 1/|\pi^2 - \lambda|$, which diverges at the first eigenvalue. As $\lambda \rightarrow 4\pi^2$, the same solution converges uniformly to $-\sin(\pi x)/(3\pi^2)$. This is only the $C = 0$ member of the infinite family at $4\pi^2$. A finite chosen limit does not remove the homogeneous freedom.