

Question 7

For $a \in \mathbb{R}$ and $L > 0$, study

$$-y'' + 2ay' = \lambda y, \quad y(0) = y(L) = 0.$$

The first derivative makes this look different from a sine eigenproblem. Use the weighted norm $\int_0^L e^{-2ax} y^2 dx$ when normalizing.

Tasks

1. Substitute $y = e^{ax}u$ and derive the transformed equation and endpoint conditions. Find every real eigenvalue and eigenspace.
2. Give the weighted unit-norm eigenfunction with positive initial derivative. Explain how the exponential factor affects shape without changing the zeros.
3. For $a = 1, L = \pi$, locate the maximum of the first eigenfunction with amplitude chosen as $y = e^x \sin x$. Sketch it and the underlying sine on the same axes.
4. Suppose the first two eigenvalues are known to be 5 and 8 but both a and L are unknown. Determine everything these data identify, and exhibit the remaining ambiguity.

Question 7 – Solution

Strategy. Remove the first derivative with an invertible exponential change and keep track of the shifted spectral parameter.

Step 1: Transform the operator completely. The derivatives are $y' = e^{ax}(u' + au)$ and $y'' = e^{ax}(u'' + 2au' + a^2u)$. Therefore

$$-y'' + 2ay' = e^{ax}(-u'' + a^2u), \quad \boxed{-u'' = (\lambda - a^2)u, \quad u(0) = u(L) = 0.}$$

The exponential is never zero, so this is a bijection of solution spaces. The shifted parameter must be positive: zero or negative gives only the zero Dirichlet solution. All eigenvalues and eigenfunctions are thus

$$\boxed{\lambda_n = a^2 + (n\pi/L)^2, \quad E_n = \text{span}\{e^{ax} \sin(n\pi x/L)\}, \quad n \geq 1.}$$

Step 2: Normalize with the correct weight. The factors e^{-2ax} and e^{2ax} cancel in the squared norm. Hence $\boxed{\phi_n = \sqrt{2/L} e^{ax} \sin(n\pi x/L)}$ has unit weighted norm and positive initial derivative. The envelope changes heights and the locations of extrema, but its positivity leaves the zeros jL/n unchanged.

Step 3: Locate the displaced maximum. For $y = e^x \sin x$, $y' = e^x(\sin x + \cos x)$ is positive up to $3\pi/4$ and negative afterwards on $(0, \pi)$. Thus

$$\boxed{x_{\max} = 3\pi/4, \quad y_{\max} = e^{3\pi/4}/\sqrt{2}.}$$

The sine itself peaks at $\pi/2$; multiplying by the increasing envelope shifts the peak without moving either zero.

Step 4: Solve the inverse spectral data. The gap $8 - 5 = 3$ equals $3\pi^2/L^2$, so $L = \pi$. Then $a^2 + 1 = 5$, giving $\boxed{a = 2 \text{ or } a = -2}$. Both choices give the entire same spectrum $4 + n^2$, yet their exponential envelopes point in opposite directions. The eigenvalues alone cannot recover the sign of the drift parameter.

