

**Question 6**

A variable-coefficient eigenproblem uses the positive interval  $[1, e^2]$ :

$$-(xy')' = \lambda \frac{y}{x}, \quad y(1) = y(e^2) = 0.$$

The natural squared norm for this problem is  $\int_1^{e^2} y(x)^2 dx/x$ .

**Tasks**

1. Use  $t = \log x$  and  $y(x) = u(t)$  to transform the equation and both boundary conditions. Show every derivative factor.
2. Find all real eigenvalues and eigenspaces, checking all parameter signs. Normalize each mode to have weighted norm one and  $y'(1) > 0$ .
3. Locate the first mode's maximum and all interior zeros of the  $n$ th mode. Explain why equal spacing in the transformed coordinate does not imply equal spacing in  $x$ .
4. Replace  $e^2$  by a general right endpoint  $b > 1$ . Give the eigenvalues and weighted normalization factor as functions of  $b$ . Sketch the first two normalized modes for the original interval.

## Question 6 – Solution

**Strategy.** Transform the differential equation and its weight together, not just its apparent frequency.

**Step 1: Carry out the logarithmic substitution.** Since  $dt/dx = 1/x$ ,  $y' = u'/x$ ,  $xy' = u'$  and  $(xy')' = u''/x$ . Multiplying the equation by  $x$  gives

$$-u'' = \lambda u, \quad u(0) = u(2) = 0.$$

Also  $dx/x = dt$ , so the weighted squared norm is exactly  $\int_0^2 u^2 dt$ .

**Step 2: Classify and normalize.** At zero the affine function with two zero endpoints is zero. At a negative parameter, the left-zero hyperbolic sine cannot vanish at the right endpoint. For  $\lambda = k^2 > 0$ ,  $u = A \sin(kt)$  and  $\sin(2k) = 0$ . Thus

$$\lambda_n = (n\pi/2)^2, \quad \phi_n(x) = \sin\left(\frac{n\pi \log x}{2}\right), \quad n \geq 1.$$

The eigenspace is the span of  $\phi_n$ . Its weighted squared norm is one, and  $\phi'_n(1) = n\pi/2 > 0$ , fixing both amplitude and sign.

**Step 3: Read locations in the original coordinate.** For  $n = 1$ , the sine attains its unique maximum one when  $t = 1$ , hence  $x = e$ , the geometric rather than arithmetic midpoint of the endpoints. For mode  $n$ , the interior zeros are  $x_j = e^{2j/n}$ ,  $j = 1, \dots, n-1$ . They form a geometric progression: logarithmic spacing becomes multiplicative spacing in  $x$ .

**Step 4: Change the endpoint.** For  $[1, b]$  the transformed length is  $\ell = \log b > 0$ . Consequently

$$\lambda_n = (n\pi/\log b)^2, \quad \phi_n = \sqrt{\frac{2}{\log b}} \sin\left(\frac{n\pi \log x}{\log b}\right).$$

The graph for  $b = e^2$  uses the physical coordinate  $x$ , with the landmark  $x = e$ .

