

Question 5

Consider a ring represented by the interval $[0, 2\pi]$ with matching values and slopes at its ends:

$$-y'' = \lambda y, \quad y(0) = y(2\pi), \quad y'(0) = y'(2\pi).$$

Here the eigenspace dimension need not be one. No Fourier expansion is required.

Tasks

1. Rule out negative eigenvalues using the boundary terms in an energy identity, and determine the zero eigenspace directly.
2. Find all positive eigenvalues and the dimension of each eigenspace. Explain why imposing one endpoint equation alone would be insufficient.
3. At a positive eigenvalue n^2 , impose $y(0) = 0$ and $y'(0) = 1$. Find the resulting eigenfunction and decide whether these normalization data are possible at zero.
4. For the normalized positive modes, test the additional measurements $y(\pi) = 0$ and $y'(\pi) = y'(0)$. Which measurement adds no information, and which selects only part of the spectrum?

Question 5 – Solution

Strategy. Treat value and derivative matching together as a two-dimensional endpoint map.

Step 1: Handle nonpositive parameters. The boundary term $[yy']_0^{2\pi}$ is zero because both values and slopes match. Thus $\lambda \int y^2 = \int (y')^2 \geq 0$. At zero, $y = A + Bx$; value matching forces $B = 0$ and slope matching is then automatic. Hence $E_0 = \text{span}\{1\}$, of dimension one.

Step 2: Find the positive eigenspaces. For $\lambda = k^2 > 0$, write $y = A \cos(kx) + B \sin(kx)$. With $\theta = 2\pi k$, the two conditions are

$$\begin{pmatrix} \cos \theta - 1 & \sin \theta \\ -\sin \theta & \cos \theta - 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The determinant is $2(1 - \cos \theta)$, so nonzero solutions require $k = n$, a positive integer. Then the entire matrix is zero and both constants are free:

$$\lambda_n = n^2, \quad E_n = \text{span}\{\cos nx, \sin nx\}, \quad n \geq 1.$$

Each positive eigenspace has dimension two. One equation alone would generally leave a nonzero vector even when the other endpoint equation fails.

Step 3: Apply the initial normalization. The data $y(0) = 0, y'(0) = 1$ require $A = 0, nB = 1$, giving $y_n = \sin(nx)/n$. At zero every eigenfunction is constant, so its initial derivative cannot be one. These extra data select a representative from each positive eigenspace, not from the zero eigenspace.

Step 4: Interpret the midpoint measurements. For every positive integer n , $y_n(\pi) = \sin(n\pi)/n = 0$, so that measurement adds no restriction. But $y'_n(\pi) = \cos(n\pi) = (-1)^n$ equals $y'_n(0) = 1$ exactly for even n . The second measurement selects eigenvalues $4, 16, 36, \dots$ from the original positive spectrum.