

Question 4

Let $h \in \mathbb{R}$ control the endpoint condition

$$-y'' = \lambda y, \quad 0 \leq x \leq 1, \quad y(0) = 0, \quad y'(1) = h y(1).$$

Unlike fixed zero endpoint values, this boundary condition can permit a negative eigenvalue.

Tasks

1. Determine precisely when zero is an eigenvalue and identify its eigenspace.
2. For $\lambda = -\mu^2 < 0$, derive the equation for μ . Prove that there is exactly one negative eigenvalue when $h > 1$ and none when $h \leq 1$.
3. For $\lambda = k^2 > 0$, derive the frequency equation without losing or adding modes by division. Locate all its roots by intervals between multiples of π , including the special first interval.
4. Derive the energy identity and explain why it does not always imply nonnegative eigenvalues. For $h = 2$, bound the negative eigenvalue using $1 < \mu < 2$, and sketch the equation locating μ .

Question 4 – Solution

Strategy. Separate all three signs and prove root counts by monotonicity on the correct domains.

Step 1: Check the zero parameter directly. At $\lambda = 0$, $y = Ax$ and the right condition is $A = hA$. Thus zero is an eigenvalue exactly when $h = 1$, with eigenspace $\text{span}\{x\}$.

Step 2: Count the negative modes. For $\lambda = -\mu^2$, the nonzero solution is a multiple of $\sinh(\mu x)$, and its endpoint condition is $\mu \coth \mu = h$. The function $g(\mu) = \mu \coth \mu$ tends to one as $\mu \downarrow 0$ and to infinity as $\mu \rightarrow \infty$. Its derivative has numerator $\sinh \mu \cosh \mu - \mu > 0$, since this numerator starts at zero and has derivative $2 \sinh^2 \mu > 0$. Thus precisely $h > 1$ gives one negative mode.

Step 3: Locate every positive mode. Here $y = A \sin(kx)$ and $k \cos k = h \sin k$. A positive multiple of π cannot solve this equation, so division by $\sin k$ is safe at a root:

$$k \cot k = h.$$

Its derivative is $(\sin k \cos k - k)/\sin^2 k < 0$ for $k > 0$ away from poles. On each $(n\pi, (n+1)\pi)$, $n \geq 1$, it decreases from $+\infty$ to $-\infty$, giving exactly one root for every h . On $(0, \pi)$ it decreases from the unattained limit one to $-\infty$: one root if $h < 1$, none if $h \geq 1$.

Step 4: Retain the endpoint energy term. Integration by parts gives $\lambda \int_0^1 y^2 dx = \int_0^1 (y')^2 dx - hy(1)^2$, which can be negative. For $h = 2$, $g(1) < 2 < g(2)$: the first inequality follows from $e^2 > 3$ and $g(2) > 2$ follows from $\coth 2 > 1$. Thus $1 < \mu < 2$ and $-4 < \lambda_- < -1$. The graph locates the unique crossing.

