

Question 2

Study the derivative boundary conditions

$$-y'' = \lambda y, \quad 0 \leq x \leq L, \quad y'(0) = y'(L) = 0, \quad L > 0.$$

A calculation that divides by $\sqrt{\lambda}$ can lose an important mode.

Tasks

1. Find every real eigenvalue and eigenspace, including a separate check at zero and an exclusion of negative eigenvalues.
2. Normalize each mode by $\int_0^L y^2 dx = 1$ and $y(0) > 0$. Explain why the zero mode has a different normalization factor.
3. Integrate the differential equation to prove that every nonzero-eigenvalue mode has zero mean. Decide what happens to the zero eigenspace if zero mean is imposed as an extra constraint.
4. For $L = 2$, test the claims that x is a zero-mode eigenfunction and that $1 + \cos(\pi x/2)$ is an eigenfunction. Distinguish satisfying both endpoint conditions from belonging to a single eigenspace.

Question 2 – Solution

Strategy. Preserve the constant solution by checking the zero parameter before dividing by a frequency.

Step 1: Find the complete spectrum. Integration by parts for any eigenfunction gives $\lambda \int_0^L y^2 dx = \int_0^L (y')^2 dx \geq 0$, so negative eigenvalues are impossible. At zero, $y = A + Bx$ and the derivative conditions force $B = 0$; nonzero constants are eigenfunctions. For $\lambda = k^2 > 0$, $y = A \cos(kx) + B \sin(kx)$; the left derivative condition forces $B = 0$ and the right requires $\sin(kL) = 0$. Thus

$$\lambda_n = (n\pi/L)^2, \quad E_n = \text{span}\{\cos(n\pi x/L)\}, \quad n = 0, 1, 2, \dots$$

Step 2: Normalize the modes. The constant has squared integral L , while every positive-frequency cosine has squared integral $L/2$. The positive value at zero fixes the signs:

$$\phi_0 = 1/\sqrt{L}, \quad \phi_n = \sqrt{2/L} \cos(n\pi x/L) \quad (n \geq 1).$$

Step 3: Interpret the mean constraint. Integrating gives $\lambda \int_0^L y dx = -y'(L) + y'(0) = 0$. Thus all modes with $\lambda \neq 0$ have zero mean. The only constant with zero mean is the zero function, so the additional constraint removes the zero eigenspace from the admissible nonzero functions; it preserves every higher mode.

Step 4: Test the two proposed functions. The function x has derivative one at both ends, so it fails the boundary conditions. The function $g = 1 + \cos(\pi x/2)$ satisfies both derivative conditions, but $-g'' = (\pi^2/4) \cos(\pi x/2)$ is not any constant multiple of g . Indeed, matching the constant term would require $\lambda = 0$, which fails for the cosine term. A sum from different eigenspaces need not be an eigenfunction.