

Question 10

A positive point load acts at an unknown location $a \in (0, 1)$:

$$-y'' = F\delta(x - a), \quad y(0) = y(1) = 0, \quad F > 0.$$

Seek a continuous, piecewise linear displacement. The distributional equation requires $y'(a+) - y'(a-) = -F$; no jump in y is allowed.

Tasks

1. Derive the solution on each side of a using the endpoint values, continuity and derivative jump. Verify the distributional equation.
2. Find the peak height and the total area $\int_0^1 y(x) dx$. For fixed F , determine which load location maximizes the height.
3. Suppose only the peak height and total area are measured, while both F and a are unknown. Decide whether these measurements determine the load uniquely, and characterize all compatible loads for a positive measured height H .
4. Instead measure the endpoint slopes $m_L = y'(0)$ and $m_R = y'(1)$. Recover F, a and state their admissibility conditions. Solve the case $m_L = 2, m_R = -1$, and sketch it alongside the reflected load at $1 - a$ with the same F .

Question 10 – Solution

Strategy. Use the boundary conditions to determine two lines, then distinguish independent measurements from redundant ones.

Step 1: Match the two linear pieces. Write $y = mx$ to the left and $y = n(1-x)$ to the right. Continuity requires $ma = n(1-a)$, and the slope jump gives $-n - m = -F$. Thus

$$y(x) = \begin{cases} F(1-a)x, & 0 \leq x \leq a, \\ Fa(1-x), & a \leq x \leq 1. \end{cases}$$

Both endpoint values vanish. The continuous join produces no δ' term; the derivative jump $-F$ gives exactly $-y'' = F\delta(x-a)$, with zero ordinary second derivative elsewhere.

Step 2: Compute the peak and area. The left slope is positive and the right slope negative, so the peak is at a :

$$H = Fa(1-a), \quad \int_0^1 y \, dx = H/2.$$

The area is that of a triangle with base one and height H . For fixed F , $a(1-a) = 1/4 - (a - 1/2)^2$ is maximized at $a = 1/2$, giving height $F/4$.

Step 3: Test whether the measurements are independent. If the measured area differs from $H/2$, no load fits the model. If it equals $H/2$, every $a \in (0, 1)$ is compatible with $\boxed{F = H/[a(1-a)]}$. The two measurements supply only one independent number, so there are infinitely many compatible location-strength pairs.

Step 4: Recover the load from endpoint slopes. Since $m_L = F(1-a)$ and $m_R = -Fa$,

$$F = m_L - m_R, \quad a = \frac{-m_R}{m_L - m_R}.$$

The precise admissibility conditions are $m_L > 0$ and $m_R < 0$. For $(m_L, m_R) = (2, -1)$, $F = 3$, $a = 1/3$, with height $2/3$ and area $1/3$. The reflected load has the same height and area but different endpoint slopes.

