

Question 9

Let $L[y] = -y'' + q(x)y$ on $[0, 1]$, where $q \geq 0$ is continuous. All solutions below satisfy $y(0) = y(1) = 0$. A useful comparison statement is: if $L[w] \geq 0$ and $w(0), w(1) \geq 0$, then $w \geq 0$ on the interval.

Tasks

1. Prove that $L[y] = f$ has exactly one solution for every continuous f . Use an energy identity for uniqueness and a family of initial-value solutions for existence.
2. Prove the stated comparison result. To handle the case where q can vanish, apply the interior-minimum test first to $w + \eta[1 + x(1 - x)]$ with $\eta > 0$, then let $\eta \downarrow 0$.
3. If an approximation v has zero endpoint values and $|L[v] - f| \leq \epsilon$, prove a pointwise error bound using $\phi = x(1 - x)/2$. Give a sharp constant for the uniform error estimate over all such problems.
4. Apply the result to $q = x^2$, $f = 1$, $v = \phi$. Compute the maximum residual exactly and give rigorous upper and lower bounds for the unknown exact solution, without claiming the approximation solves the equation.

Question 9 – Solution

Strategy. Combine an energy identity, a strict comparison argument and a simple barrier to certify the boundary-value error.

Step 1: Establish existence and uniqueness. A homogeneous difference w with zero endpoint values satisfies $0 = \int_0^1 wL[w]dx = \int_0^1 ((w')^2 + qw^2)dx$, hence $w' = 0$ and $w = 0$. For existence, let p solve $L[p] = f$ with $p(0) = p'(0) = 0$, and let h solve $L[h] = 0$ with $h(0) = 0, h'(0) = 1$. Regular IVP theory supplies both. If $h(1) = 0$, uniqueness just proved would force $h = 0$, a contradiction. Thus $y = p - p(1)h/h(1)$ is the required solution.

Step 2: Make comparison strict before taking a limit. Put $\psi = 1 + x(1 - x) > 0$. Then $L[\psi] = 2 + q\psi > 0$, so $L[w + \eta\psi] > 0$ and its endpoint values are positive. At a nonpositive interior minimum, its second derivative is nonnegative and $q \geq 0$ would give $L[w + \eta\psi] \leq 0$, a contradiction. Thus $w + \eta\psi > 0$. Letting $\eta \downarrow 0$ proves $w \geq 0$.

Step 3: Bound the error with a barrier. Since $L[\phi] = 1 + q\phi \geq 1$, for $e = y - v$ we have $L[\epsilon\phi \pm e] \geq 0$, with zero endpoints. Comparison yields

$$|y(x) - v(x)| \leq \epsilon\phi(x), \quad \max_{[0,1]} |y - v| \leq \epsilon/8.$$

The constant $1/8$ is sharp: take $q = 0, v = 0, f = \epsilon$, for which $y = \epsilon\phi$ attains equality at $x = 1/2$.

Step 4: Certify the proposed approximation. Here $L[\phi] - 1 = x^2\phi = x^3(1 - x)/2$. Its derivative is $x^2(3 - 4x)/2$, so its maximum is $\epsilon = 27/512$ at $x = 3/4$. Also $L[v - y] \geq 0$, hence $y \leq v$. Combining this sign with the error bound gives

$$\left(1 - \frac{27}{512}\right)\phi \leq y \leq \phi, \quad \max_{[0,1]} |y - \phi| \leq \frac{27}{4096}.$$

The residual is not zero; these are certified bounds, not an exact formula for y .