

Question 8

A linear differential equation is paired with a nonlinear boundary condition:

$$y'' = 0, \quad y(0) = 0, \quad y'(1) = y(1)^2 + \lambda,$$

where $\lambda \in \mathbb{R}$.

Tasks

1. Reduce the boundary-value problem to an algebraic equation and classify the number of real solutions for every λ .
2. At $\lambda = 0$, test whether the average of the two solutions is also a solution. Explain which part of the full problem prevents superposition.
3. Set $\lambda = 1/4 - \varepsilon$ with $\varepsilon > 0$. Measure the uniform distances of the two solutions from the unique critical solution at $\lambda = 1/4$. Decide whether a local Lipschitz estimate in the parameter can hold there.
4. Plot the admissible slopes $c = y'(0)$ against λ , clearly indicating the critical point and where no real solution exists. Explain why uniqueness at a single parameter need not imply stable dependence on nearby parameters.

Question 8 – Solution

Strategy. Solve the differential equation completely, then retain the nonlinear structure of its endpoint constraint.

Step 1: Classify the slope equation. Every solution of $y'' = 0, y(0) = 0$ is $y = cx$. Its remaining boundary condition is $c = c^2 + \lambda$, so

$$c_{\pm} = \frac{1 \pm \sqrt{1 - 4\lambda}}{2}.$$

There are two real solutions for $\lambda < 1/4$, one for $\lambda = 1/4$, and none for $\lambda > 1/4$. The critical solution is $y_* = x/2$.

Step 2: Check the superposition claim. At $\lambda = 0$, the solutions are 0 and x . Their average $x/2$ has $y'(1) = 1/2$ but $y(1)^2 = 1/4$, so it fails the boundary condition. The differential equation is linear, but the square in the boundary operator makes the full boundary-value problem nonlinear.

Step 3: Quantify critical sensitivity. At $\lambda = 1/4 - \varepsilon$, the slopes are $1/2 \pm \sqrt{\varepsilon}$. Hence

$$\max_{[0,1]} |y_{\pm} - y_*| = \sqrt{\varepsilon}.$$

A Lipschitz bound with fixed constant K would require $\sqrt{\varepsilon} \leq K\varepsilon$ for all sufficiently small positive ε , which is impossible. Both branches approach y_* continuously from this side, but not at a rate bounded linearly by the parameter error.

Step 4: Read the complete branch picture. The branches meet at $(\lambda, c) = (1/4, 1/2)$ and have no real continuation for larger λ . For $\lambda < 1/4$, $dc_{\pm}/d\lambda = \mp 1/\sqrt{1 - 4\lambda}$, whose magnitude diverges at the join. Critical uniqueness therefore does not guarantee a locally unique family or a Lipschitz-stable response.

