

Question 7

Consider a singular-endpoint problem on $0 < x \leq 1$:

$$(x^2 y')' = 0, \quad y(1) = 1.$$

An additional condition at zero is to be imposed through a limit or regularity requirement. The differential equation is initially required only for $x > 0$.

Tasks

1. Find every solution on $(0, 1]$ satisfying the right boundary condition. Identify why zero is not a regular endpoint of the normalized equation.
2. Select the solutions that remain bounded as $x \downarrow 0$, and separately those with finite weighted energy $\int_0^1 x^2 (y')^2 dx$. Determine whether these two selections agree.
3. Replace boundedness by $\lim_{x \downarrow 0} xy(x) = k$, with k prescribed. Find the unique solution for every real k and its limiting weighted derivative $\lim_{x \downarrow 0} x^2 y'(x)$.
4. Decide whether a continuous solution can satisfy $y(0) = 0$ and $y(1) = 1$. Explain why two stated endpoint values do not justify invoking the usual regular boundary-value theory at this singular endpoint.

Question 7 – Solution

Strategy. Solve on the open interval first, then apply each proposed endpoint condition to the possible singular term.

Step 1: Find the family on the actual domain. Integration gives $x^2 y' = C$ and $y = A - C/x$. Incorporating $y(1) = 1$, write

$$y = 1 - k + k/x, \quad k \in \mathbb{R}.$$

For $x > 0$ the normalized equation is $y'' + (2/x)y' = 0$. Its coefficient $2/x$ is singular at zero; the original leading coefficient also vanishes there.

Step 2: Test boundedness and energy. Boundedness at zero forces $k = 0$, giving $y \equiv 1$. Since $y' = -k/x^2$,

$$\int_0^1 x^2 (y')^2 dx = \lim_{t \downarrow 0} k^2 \left(\frac{1}{t} - 1 \right).$$

This is infinite for $k \neq 0$ and zero for $k = 0$. Thus boundedness and finite weighted energy select the same single solution in this family.

Step 3: Prescribe a weighted endpoint value. For the displayed family,

$$\lim_{x \downarrow 0} xy(x) = k, \quad \lim_{x \downarrow 0} x^2 y'(x) = -k.$$

Therefore each real prescribed weighted limit selects exactly one solution, including unbounded solutions when $k \neq 0$. A weighted boundary condition is not the same as an ordinary finite value $y(0)$.

Step 4: Test the proposed continuous boundary data. A continuous extension is bounded, so it must be $y \equiv 1$ and must have $y(0) = 1$. Consequently no continuous solution satisfies $y(0) = 0, y(1) = 1$. One cannot assign an unrelated value at zero and retain continuity. Regular endpoint theory requires hypotheses on the normalized coefficients that fail here; the explicit family and its limits decide this problem instead.