

Question 6

A third-order model is to satisfy four endpoint measurements:

$$y''' = q, \quad y(0) = y'(0) = 0, \quad y(1) = a, \quad y'(1) = b.$$

The constant load q may either be prescribed or be an unknown to infer.

Tasks

1. For a fixed prescribed q , derive the necessary and sufficient relation between a, b for a solution to exist. Determine the number of compatible solutions.
2. Instead regard q as unknown. Recover both the load and the complete profile uniquely for arbitrary real a, b .
3. Apply the inverse problem to $a = 1, b = 0$. Verify the load and all four endpoint conditions, and prove that the resulting displacement is increasing on $(0, 1)$.
4. For a nominal load $q = 6$ and right displacement $a = 1$, determine the required right slope. Explain why the fourth boundary condition can be inconsistent when q is fixed but informative when q is unknown.

Question 6 – Solution

Strategy. Count the integration constants together with any unknown model parameter, and then solve the actual endpoint equations.

Step 1: Find compatibility for a fixed load. The left conditions give $y = qx^3/6 + cx^2/2$. The right conditions require $a = q/6 + c/2$ and $b = q/2 + c$. Subtraction yields

$$b - 2a = q/6.$$

This condition is necessary and sufficient: if it holds, $c = 2a - q/3$ gives the unique solution; otherwise no solution satisfies all the prescribed data.

Step 2: Recover an unknown load. When q is free, the same two equations determine both q and c :

$$q = 6(b - 2a), \quad y = (b - 2a)x^3 + (3a - b)x^2.$$

There is one solution for every pair (a, b) . Direct differentiation verifies the constant load and the two endpoint slopes and values.

Step 3: Check the displacement from rest to rest. For $a = 1, b = 0$,

$$q = -12, \quad y = 3x^2 - 2x^3.$$

Its third derivative is -12 , its endpoint values are zero and one, and $y' = 6x(1 - x)$ vanishes at both ends. Since this derivative is positive inside, the displacement is strictly increasing on $(0, 1)$.

Step 4: Interpret the extra measurement. For $q = 6, a = 1$, compatibility requires $b - 2 = 1$, so $b = 3$ and $y = x^3$. Any other prescribed right slope is inconsistent with those fixed data. The left conditions already remove two of the three integration constants; with fixed q , the remaining constant cannot fit arbitrary two right data. When q is unknown, it supplies one more adjustable quantity, which the fourth measurement identifies. Counting conditions is useful only with this distinction.