

Question 5

A dimensionless fourth-order beam model has constant load $y^{(4)} = 24$ on $[0, 1]$. Compare two endpoint designs:

$$\begin{array}{ll} \text{Clamped:} & y(0) = y'(0) = y(1) = y'(1) = 0, \\ \text{Simply supported:} & y(0) = y''(0) = y(1) = y''(1) = 0. \end{array}$$

Tasks

1. Find the solution for each design by integrating the fourth-order equation. Verify all four conditions separately for each solution.
2. Prove uniqueness for both designs by applying integration by parts to the difference of two solutions. Track the boundary terms explicitly.
3. Find each maximum displacement and its location. Explain why symmetry alone would not be a complete proof that the midpoint is a maximum.
4. Compare the two profiles pointwise and compute the ratio of their maximum displacements. Sketch both on the same axes and explain why changing derivative conditions changes the response under the same load.

Question 5 – Solution

Strategy. Determine the integration constants from the correct derivative data, then use an energy identity to check uniqueness.

Step 1: Solve both sets of endpoint equations. Starting from $y = x^4 + ax^3 + bx^2 + cx + d$, the two solutions are

$$y_C = x^2(1-x)^2, \quad y_S = x^4 - 2x^3 + x.$$

Both fourth derivatives equal 24 and both endpoint values are zero. Also $y'_C = 2x(1-x)(1-2x)$ vanishes at both ends, while $y'_S = 12x(x-1)$ vanishes at both ends. These verify the distinct boundary data.

Step 2: Prove uniqueness in each case. A difference w has $w^{(4)} = 0$ and homogeneous boundary conditions. Twice integrating by parts gives

$$0 = \int_0^1 ww^{(4)} dx = [ww''' - w'w'']_0^1 + \int_0^1 (w'')^2 dx.$$

The boundary term is zero for clamped data because $w = w' = 0$, and for simply supported data because $w = w'' = 0$. Thus $w'' = 0$; its zero endpoint values imply $w = 0$. Each design has exactly one solution.

Step 3: Locate and compare the maxima. The derivative of y_C changes from positive to negative at $1/2$. For the other design, $y'_S = (2x-1)(2x^2-2x-1)$, whose second factor is negative on $[0, 1]$. Thus both have their unique maximum at $1/2$, with

$$y_C(1/2) = 1/16, \quad y_S(1/2) = 5/16.$$

The derivative signs establish maxima; symmetry by itself only suggests a stationary midpoint.

Step 4: Compare the complete profiles. The difference is $y_S - y_C = x(1-x) > 0$ inside the interval, and the maximum ratio is five. The same differential load is compatible with different responses because fixing slopes differs from fixing second derivatives at the boundaries.

