

Question 4

A mixed boundary condition couples endpoint value and slope:

$$y'' = 2, \quad y(0) = 0, \quad y'(1) - h y(1) = b,$$

where h, b are real parameters.

Tasks

1. Solve the problem for $h \neq 1$ and verify all data. Classify the number of solutions at $h = 1$ for every b .
2. For fixed $h \neq 1$, determine the exact response to a perturbation $b \mapsto b + \delta$. Find its uniform size on $[0, 1]$.
3. Follow the path $b = 1$ as $h \rightarrow 1$. Explain how a well-defined limit along this path coexists with nonuniqueness at the limiting parameters.
4. For arbitrary real k , use $h_n = 1 - 1/n$ and $b_n = 1 + k/n$ to find the limiting solution. Decide whether the boundary-value solution extends continuously as a single-valued function of (h, b) at $(1, 1)$.

Question 4 – Solution

Strategy. Inspect the denominator in the endpoint equation and distinguish a chosen parameter path from a unique limiting response.

Step 1: Solve and classify the singular case. The left boundary gives $y = x^2 + cx$. The mixed condition becomes $(1 - h)c = b - 2 + h$. For $h \neq 1$,

$$y = x^2 + \left(\frac{b-1}{1-h} - 1 \right) x.$$

Its second derivative is two, its left value is zero, and substitution verifies the mixed condition. At $h = 1$, the condition reduces to $b = 1$ independently of c : infinitely many solutions if $b = 1$, and none otherwise.

Step 2: Quantify sensitivity. At fixed $h \neq 1$, the change in the response is

$$\Delta y(x) = \frac{\delta x}{1-h}, \quad \max_{[0,1]} |\Delta y| = \frac{|\delta|}{|1-h|}.$$

Thus arbitrarily small boundary errors can be strongly amplified near $h = 1$.

Step 3: Examine one compatible path. If $b = 1$ and $h \neq 1$, then $c = -1$, so every solution on this path is $y = x^2 - x$. Its limit is one member of the infinite family at $(h, b) = (1, 1)$. The path selects this member; the limiting boundary conditions alone do not.

Step 4: Compare different approaches. For the proposed sequences, $(b_n - 1)/(1 - h_n) = k$, so

$$y_n = x^2 + (k - 1)x \quad \text{for every } n.$$

All parameter pairs approach $(1, 1)$, but their solution limits depend on k . They range over the entire limiting family. Consequently no choice of a single solution at $(1, 1)$ can make the solution map continuous there along all paths.