

Question 1

A boundary-value problem prescribes values at two different points:

$$y'' = -6x, \quad y(0) = 1, \quad y(1) = 2.$$

Compare it with the family of initial-value problems having $y(0) = 1$ and $y'(0) = s$, where s is a freely adjustable initial slope.

Tasks

1. Solve the initial-value family and determine the unique slope that meets the right boundary condition. Explain the distinction between shooting for an endpoint and prescribing an initial slope.
2. Verify the resulting differential equation and both endpoint values. Find the maximum and minimum on $[0, 1]$ and determine whether endpoint values bound the entire response.
3. Replace the boundary values by arbitrary α, β . Prove existence and uniqueness for every pair and give the solution explicitly.
4. If the endpoint values change by δ_0, δ_1 while the forcing stays fixed, find the exact change in the solution and a sharp uniform bound. Sketch the original solution and the line joining its endpoints.

Question 1 – Solution

Strategy. Solve the initial-value family first, then use the endpoint equation to select its slope.

Step 1: Select the shooting slope. Integrating twice gives $y_s = 1 + sx - x^3$. The right endpoint is $y_s(1) = s$, so $s = 2$ is the unique successful slope. An initial-value problem fixes s directly; the boundary-value problem determines it from a condition at another point.

Step 2: Verify the solution and its extrema. The solution $y = 1 + 2x - x^3$ has $y'' = -6x$, $y(0) = 1$, $y(1) = 2$. Since $y' = 2 - 3x^2$, its maximum occurs at $r = \sqrt{2/3}$:

$$y_{\max} = 1 + \frac{4}{3}\sqrt{\frac{2}{3}} > 2, \quad y_{\min} = y(0) = 1.$$

Thus the endpoint values do not bound this forced response from above.

Step 3: Allow arbitrary endpoint values. Direct integration gives

$$y_{\alpha,\beta}(x) = \alpha + (\beta - \alpha + 1)x - x^3.$$

This works for every α, β . The difference of two solutions with the same data satisfies $w'' = 0$, $w(0) = w(1) = 0$, so it is zero, proving uniqueness.

Step 4: Quantify boundary-data sensitivity. The exact response change is $\Delta y = (1 - x)\delta_0 + x\delta_1$. Therefore

$$\max_{0 \leq x \leq 1} |\Delta y| = \max(|\delta_0|, |\delta_1|).$$

The inequality follows from convex weights; equality holds at an endpoint. The original curve exceeds its endpoint line $1 + x$ by $x - x^3 > 0$ inside $(0, 1)$.

