

Question 8

Investigate the forced singular equation

$$xy''' - y'' = x, \quad x > 0.$$

One wishes to continue a solution to $x = 0$. Analytic continuation, twice continuous differentiability and three times continuous differentiability are different requirements here.

Tasks

1. Attempt an ordinary power series at zero. Identify the precise coefficient equation that obstructs a formal Taylor solution.
2. Solve the equation on $x > 0$ by first setting $z = y''$. Give the complete three-constant family.
3. Determine the highest integer k for which the solutions extend as C^k functions to $[0, \infty)$. Explain why no choice of the homogeneous constants removes the obstruction.
4. Select the constants giving $y = x^3 \log x/6$. Find its endpoint limit, its minimum on $(0, 1]$ and its value at one. Sketch it, showing zero as an excluded endpoint of the original domain.

Question 8 – Solution

Strategy. Let the failed coefficient equation identify a logarithmic term forced by resonance.

Step 1: Locate the formal obstruction. For $y = \sum a_n x^n$, matching x^n gives

$$(n+2)(n+1)(n-1)a_{n+2} = \begin{cases} 1, & n = 1, \\ 0, & n \neq 1. \end{cases}$$

At $n = 0$ this requires $a_2 = 0$, but at $n = 1$ it requires $0 = 1$. No formal ordinary power series, and hence no analytic solution, exists at zero.

Step 2: Solve the reduced equation. For $z = y''$, $(z/x)' = 1/x$, giving $z = x \log x + Cx$. Integrating twice yields

$$y = A + Bx + \frac{x^3 \log x}{6} + \left(\frac{C}{6} - \frac{5}{36}\right)x^3.$$

The three homogeneous modes are $1, x, x^3$; the forced logarithm is additional.

Step 3: Classify endpoint smoothness. As $x \downarrow 0$, $y \rightarrow A$, $y' \rightarrow B$ and $y'' = x \log x + Cx \rightarrow 0$. These limits give a C^2 extension. But $y''' = \log x + 1 + C \rightarrow -\infty$, so no C^3 extension exists. Constants can change finite terms, not cancel the coefficient of $\log x$. This is a formal inconsistency, unlike a solvable but divergent coefficient recurrence.

Step 4: Verify the illustrated member. Set $A = B = 0$, $C = 5/6$. Then $y' = x^2(3 \log x + 1)/6$, so

$$x_{\min} = e^{-1/3}, \quad y_{\min} = -\frac{1}{18e}, \quad y(1) = 0, \quad \lim_{x \downarrow 0} y(x) = 0.$$

The derivative changes from negative to positive at this unique minimum. The open marker records the original domain $x > 0$, even though a C^2 extension to zero is possible.

