

Question 7

Use series to investigate the directional boundedness of solutions to

$$y''' - 2xy'' - 4y' = 0, \quad y(0) = a, \quad y'(0) = c, \quad y''(0) = 2b.$$

You may use $\int_0^\infty e^{-t^2} dt = \sqrt{\pi}/2$.

Tasks

1. Derive the coefficient recurrence and construct three entire basis functions normalized by the three initial coefficients.
2. Sum the even chain and express the odd chain as an integral involving e^{-t^2} . Verify the resulting general solution and its initial values.
3. Give necessary and sufficient conditions for boundedness as $x \rightarrow +\infty$, as $x \rightarrow -\infty$, and in both directions. Justify cancellation limits with an explicit Gaussian-tail bound.
4. For $a = b = -\sqrt{\pi}/2$, $c = 1$, determine the limit at $+\infty$ and the sign of the response for $x \geq 0$. Sketch that one-sided bounded response on $[0, 4]$.

Question 7 – Solution

Strategy. Build an entire basis from the recurrence, then examine its exponentially amplified coefficients at each end of the real line.

Step 1: Follow the even and odd chains. Matching x^n gives $a_{n+3} = 2a_{n+1}/(n+3)$ for $n \geq 0$. The independent seeds are $a_0 = a$, $a_1 = c$, $a_2 = b$. Ratio tests along each chain give infinite radius. A normalized basis is

$$1, \quad e^{x^2} - 1, \quad H(x) = x + \frac{2x^3}{3} + \frac{4x^5}{15} + \cdots.$$

Step 2: Identify and verify the odd basis. The function $H = e^{x^2} \int_0^x e^{-t^2} dt$ satisfies $H' = 2xH + 1$; differentiating twice proves the third-order equation and its initial triple $(0, 1, 0)$. The other two basis functions also satisfy the equation. Thus

$$y = a + b(e^{x^2} - 1) + ce^{x^2} \int_0^x e^{-t^2} dt.$$

Its initial coefficients are exactly (a, c, b) in degrees zero, one and two.

Step 3: Classify both ends. Put $K = \sqrt{\pi}/2$. At $+\infty$, a nonzero $b + cK$ forces exponential growth. If $b + cK = 0$, the remaining tail tends to zero because, for $x > 0$,

$$0 \leq e^{x^2} \int_x^\infty e^{-t^2} dt \leq \frac{1}{2x}.$$

The inequality follows by replacing 1 with t/x in the integral. Therefore boundedness at $+\infty$ is equivalent to $b + cK = 0$, with limit $a - b$. Oddness of the integral gives the condition $b - cK = 0$ at $-\infty$, also with limit $a - b$. Both conditions hold exactly when $b = c = 0$, leaving constants.

Step 4: Check the selected response. Here $y = -e^{x^2} \int_x^\infty e^{-t^2} dt < 0$ for $x \geq 0$, and the tail estimate gives $y \rightarrow 0$ as $x \rightarrow +\infty$.

