

Question 6

The series solution of $y''' + y' = 0$ with $(y, y', y'')(0) = (0, 1, 0)$ describes an oscillator derivative pair (y', y'') . A proposed rational approximation is

$$U(x) = \frac{1 - x^2/4}{1 + x^2/4}, \quad V(x) = -\frac{x}{1 + x^2/4}.$$

Its designer claims that correct initial values and exact conservation of $U^2 + V^2$ are enough to make it the true derivative pair.

Tasks

1. Derive the series recurrence and identify the exact solution. Give its terms through degree seven.
2. Expand U, V through the first coefficient at which each differs from the true pair. Check their initial values and quadratic invariant.
3. Test the necessary derivative relation $U' = V$. Compute its defect exactly and decide whether any function can have both $y' = U$ and $y'' = V$ on an interval containing zero.
4. Find the phase $\tau(x)$ for which $(U, V) = (\cos \tau, -\sin \tau)$. Explain the actual evolution law and plot the phase lag $x - \tau(x)$ beside its leading cubic approximation on $[0, 2]$.

Question 6 – Solution

Strategy. Test derivative compatibility in addition to an invariant and a few Taylor coefficients.

Step 1: Solve the series IVP. Coefficient matching gives $a_{n+3} = -a_{n+1}/[(n+3)(n+2)]$. The data $(a_0, a_1, a_2) = (0, 1, 0)$ give the entire series

$$y = \sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots$$

The true pair is $(\cos x, -\sin x)$.

Step 2: Check local accuracy and the invariant. Near zero,

$$U = 1 - x^2/2 + x^4/8 + O(x^6), \quad V = -x + x^3/4 + O(x^5).$$

The correct coefficients are $x^4/24$ and $x^3/6$, respectively. Nevertheless $(U, V)(0) = (1, 0)$, and direct squaring gives $U^2 + V^2 = 1$ exactly.

Step 3: Expose the derivative defect. Writing $d = 1 + x^2/4$, we obtain

$$U' - V = \frac{x^3}{4d^2}.$$

This is nonzero for $x \neq 0$. If $y' = U$, differentiation forces $y'' = U'$, so no such function has both proposed derivatives on a nontrivial interval containing zero. Conservation alone does not enforce the differential equation.

Step 4: Identify the altered clock. The half-angle identities give $\tau = 2 \arctan(x/2)$ and $\tau' = 1/d$. Thus $U' = V/d$, $V' = -U/d$: the pair traverses the correct circle with a different speed. The phase lag is $x - \tau = x^3/12 + O(x^5)$ and is positive for $x > 0$ because its derivative is $x^2/(4 + x^2) > 0$.

