

Question 5

A formal power series is an algebraic coefficient expansion without an assumed positive radius of convergence. Investigate

$$x^2 y''' - y'' = -x, \quad y(0) = y'(0) = y''(0) = 0.$$

For a possible one-sided smooth solution, you may use

$$z(x) = \int_0^\infty e^{-t} \frac{x}{1 - xt} dt, \quad x \leq 0.$$

Tasks

1. Put $z = y''$ and derive its formal coefficient recurrence. Find the first four nonzero terms of the formal series for y .
2. Prove that the formal series for y has radius zero. Deduce whether an analytic solution at zero can satisfy the equation and data.
3. Verify that the given integral defines a smooth function from the left at zero and satisfies $x^2 z' - z = -x$. Recover a one-sided smooth solution y with the required initial data.
4. Explain how this smooth solution can have the divergent formal series as its Taylor series. Identify the hypothesis that prevents application of the ordinary-point analytic existence theorem at zero.

Question 5 – Solution

Strategy. Separate formal coefficient solvability from convergence and from one-sided smooth existence.

Step 1: Compute the formal coefficients. For $z = \sum b_n x^n$, coefficient matching gives $b_0 = 0$, $b_1 = 1$, and $b_n = (n-1)b_{n-1}$ for $n \geq 2$. Thus $b_n = (n-1)!$ for $n \geq 1$. Two integrations with zero constants give

$$y \sim \sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)(n+2)} x^{n+2} = \frac{x^3}{6} + \frac{x^4}{12} + \frac{x^5}{10} + \frac{x^6}{5} + \cdots.$$

Step 2: Prove divergence. The ratio of consecutive displayed coefficients is $n(n+1)/(n+3) \rightarrow \infty$. For every fixed nonzero x , the terms eventually fail to tend to zero. The radius is zero. Any analytic solution would have exactly these forced coefficients, so no analytic solution at zero exists.

Step 3: Construct and verify a smooth solution. For $x \leq 0$, $1 - xt \geq 1$. For $n \geq 1$,

$$\partial_x^n \frac{x}{1 - xt} = \frac{n! t^{n-1}}{(1 - xt)^{n+1}}.$$

An integrable majorant permits differentiation of every order up to the left endpoint, giving $z(0) = 0$ and $z^{(n)}(0-) = n!(n-1)!$. Also

$$x^2 z' - z = x \int_0^\infty \frac{d}{dt} \left(\frac{e^{-t}}{1 - xt} \right) dt = -x.$$

Hence $y(x) = \int_0^x (x-v)z(v) dv$, with oriented integration, is smooth on $(-\infty, 0]$, satisfies the equation there and has the zero initial triple. The equality at zero follows by continuity.

Step 4: Resolve the apparent contradiction. Its derivatives give exactly the formal coefficients above, but smoothness does not require the Taylor series to converge or represent the function. The leading coefficient x^2 vanishes at zero, so this is not an ordinary point. The analytic existence theorem for a normalized regular equation does not apply.