

Question 2

For the fourth-order problem

$$y^{(4)} + y'' + y = e^x, \quad y^{(j)}(0) = 0 \quad (0 \leq j \leq 3),$$

let $b_n = y^{(n)}(0)$ and $a_n = b_n/n!$. The ordinary generating function $B(z) = \sum_{n \geq 0} b_n z^n$ is not the solution's Taylor series.

Tasks

1. Derive a recurrence for b_n and prove that the derivative sequence is periodic. Give one full period.
2. Find $B(z)$ and its radius of convergence. Write the actual solution series using a_n and determine its radius separately.
3. Verify the differential equation directly from the solution series and explain why treating b_n as ordinary Taylor coefficients would give a wrong answer.
4. Determine $\lim_{x \rightarrow +\infty} e^{-x}y(x)$ without explicitly solving for all homogeneous constants. Justify why the homogeneous contribution disappears in this limit.

Question 2 – Solution

Strategy. Keep derivative values separate from coefficients divided by factorials.

Step 1: Derive periodic derivative data. Differentiating the equation n times at zero gives

$$b_{n+4} + b_{n+2} + b_n = 1 \quad (n \geq 0).$$

The first six values are $(0, 0, 0, 0, 1, 1)$. Subtracting this equation from its version at $n + 2$ gives $b_{n+6} = b_n$. Thus $b_n = 1$ precisely when $n \equiv 4, 5 \pmod{6}$, and otherwise $b_n = 0$.

Step 2: Compare the two generating functions. The ordinary generating function of the derivatives is

$$B(z) = \frac{z^4 + z^5}{1 - z^6}, \quad R_B = 1.$$

There is a genuine pole at $z = 1$. In contrast, the solution is

$$y(x) = \sum_{k=0}^{\infty} \left(\frac{x^{6k+4}}{(6k+4)!} + \frac{x^{6k+5}}{(6k+5)!} \right), \quad R_y = \infty.$$

Absolute convergence follows by comparison with $e^{|x|}$.

Step 3: Verify the equation and normalization. The coefficient of $x^n/n!$ in $y^{(4)} + y'' + y$ is $b_{n+4} + b_{n+2} + b_n = 1$, giving e^x . The first four derivative values vanish. Using b_n as ordinary coefficients would make the fourth derivative at zero 24, whereas the equation and initial values require $y^{(4)}(0) = 1$.

Step 4: Find the dominant exponential. A particular solution is $e^x/3$, since $1^4 + 1^2 + 1 = 3$. The characteristic polynomial factors as

$$r^4 + r^2 + 1 = (r^2 + r + 1)(r^2 - r + 1).$$

Its distinct roots have real parts $-1/2$ or $1/2$. Every homogeneous term is therefore $O(e^{x/2})$ as $x \rightarrow +\infty$, regardless of its constants. It follows that

$$\lim_{x \rightarrow +\infty} e^{-x} y(x) = 1/3.$$