

Question 1

Consider the entire-coefficient problem

$$y^{(4)} = x^2 y'' + 2xy', \quad (y, y', y'', y''')(0) = (0, 1, 0, 0).$$

Write $y = \sum_{n \geq 0} a_n x^n$ using ordinary power-series coefficients.

Tasks

1. Derive the recurrence, including its exceptional first index. Identify all potentially nonzero coefficient classes.
2. Find the terms through degree thirteen and verify the initial data and the recurrence relations determining those terms.
3. Prove that the solution series is entire. For the degree-nine truncation p_9 , give a rigorous uniform error bound below 10^{-6} on $[-1, 1]$.
4. Determine the sign of $y - p_9$ on either side of zero. Plot the error on $[0, 1]$ together with a pointwise certificate, explicitly scaling the vertical values by 10^6 .

Question 1 – Solution

Strategy. Follow the surviving coefficient chain and bound its tail by a geometric majorant.

Step 1: Match every coefficient. The coefficient of x^n on the right is $n(n+1)a_n$, so

$$a_{n+4} = \frac{n a_n}{(n+4)(n+3)(n+2)} \quad (n \geq 0).$$

In particular $a_4 = 0$, not a freely chosen coefficient. The initial coefficients are $(a_0, a_1, a_2, a_3) = (0, 1, 0, 0)$; only indices $1 \pmod{4}$ survive.

Step 2: Compute the first terms. Successive use of $n = 1, 5, 9$ gives

$$y = x + \frac{x^5}{60} + \frac{x^9}{6048} + \frac{x^{13}}{1153152} + \cdots$$

The missing coefficients are zero. The first four derivatives at zero match the data, and substitution into the recurrence verifies the displayed chain.

Step 3: Prove convergence and certify the tail. For fixed x , successive nonzero terms have ratio in magnitude $|x|^4 n / [(n+4)(n+3)(n+2)] \rightarrow 0$, so the series is entire and may be differentiated termwise. For $n \geq 9$, the coefficient ratio is at most $1/(n+2)^2 < 1/100$. Consequently, for $|x| \leq 1$,

$$|y - p_9| \leq \frac{100|x|^{13}}{99 \cdot 1153152} \leq B(x) := \frac{11|x|^{13}}{10 \cdot 1153152} \leq \frac{11}{10 \cdot 1153152} < 10^{-6}.$$

This bound concerns the full infinite tail, not just its first omitted term.

Step 4: Check the error's sign and graph. Every surviving coefficient is positive and every surviving degree is odd. Thus $y - p_9$ is positive for $x > 0$, negative for $x < 0$, and zero at zero. The figure compares $10^6(y - p_9)$ and $10^6 B(x)$ on $[0, 1]$.

