

Question 10

For the stable scalar equation $(D + 1)^3 y = 0$, introduce scaled state coordinates

$$q_1 = y, \quad q_2 = (y' + y)/4, \quad q_3 = (y'' + 2y' + y)/16.$$

Let $\|\cdot\|_2$ be the Euclidean norm and consider $t \geq 0$.

Tasks

1. Derive the constant system for q and the inverse map to (y, y', y'') . Explain why all characteristic roots are still -1 .
2. Solve the system for $q(0) = (0, 0, 1)^T$. Show explicitly that its Euclidean norm can exceed its initial value even though every solution tends to zero.
3. Set $w = (q_1, 4q_2, 16q_3)^T$. Prove a quadratic norm estimate that is strictly decreasing for nonzero states, using

$$2w_2(w_1 + w_3) \leq \sqrt{2}\|w\|_2^2.$$

Obtain an exponential bound for $\|q(t)\|_2$ valid for every initial state.

4. Explain why transient growth is consistent with asymptotic stability. For the prescribed initial state, sketch $\|q(t)\|_2$ and $\|w(t)\|_2/16$, labeling the normalization, and distinguish the two quantities.

Question 10 – Solution

Strategy. Compare Euclidean growth in scaled coordinates with decay in a suitable weighted norm.

Step 1: Derive the scaled cascade. With N having ones just above the diagonal,

$$q' = (-I + 4N)q, \quad y = q_1, \quad y' = 4q_2 - q_1, \quad y'' = 16q_3 - 8q_2 + q_1.$$

The coordinate map is invertible, so this is a similarity transformation of the scalar companion system. Its triangular matrix has three eigenvalues -1 .

Step 2: Compute an actual transient. Since $N^3 = 0$, $e^{(-I+4N)t} = e^{-t}(I + 4tN + 8t^2N^2)$. Therefore

$$q = e^{-t}(8t^2, 4t, 1)^T, \quad \|q(t)\|_2 = e^{-t}\sqrt{64t^4 + 16t^2 + 1}.$$

The initial norm is one, but $\|q(1)\|_2 = 9/e > 1$. Every state still tends to zero because its components are polynomials times e^{-t} .

Step 3: Prove a uniform weighted estimate. The new state satisfies $w' = (-I + N)w$, so

$$\frac{d}{dt}\|w\|_2^2 = -2\|w\|_2^2 + 2w_2(w_1 + w_3) \leq -(2 - \sqrt{2})\|w\|_2^2.$$

The stated inequality follows from Cauchy–Schwarz and $2\sqrt{2}|w_2|\sqrt{w_1^2 + w_3^2} \leq \sqrt{2}\|w\|_2^2$. With $\alpha = 1 - 1/\sqrt{2} > 0$, integration and $\|q\|_2 \leq \|w\|_2 \leq 16\|q\|_2$ give

$$\|q(t)\|_2 \leq 16e^{-\alpha t}\|q(0)\|_2.$$

Step 4: Interpret coordinate-dependent growth. Asymptotic stability allows a finite transient; it does not require every norm to decrease at every instant. For this trajectory, $\|w\|_2/16 = e^{-t}\sqrt{t^4/4 + t^2 + 1}$ decreases strictly, whereas $\|q\|_2$ has a pronounced transient. They measure the same state with different weights.

