

Question 9

A fourth-order oscillator has a repeated quadratic factor:

$$y^{(4)} + 2y'' + y = 0, \quad t \geq 0.$$

Introduce the state

$$W = (y, y', y'' + y, y''' + y')^T.$$

All four scalar initial derivatives are arbitrary real numbers.

Tasks

1. Derive the system for W and show that this coordinate change is invertible. Identify the autonomous oscillator inside the system.
2. Write the full solution in terms of $a = y(0)$, $b = y'(0)$, $A = y''(0) + y(0)$ and $B = y'''(0) + y'(0)$.
3. Give necessary and sufficient conditions on the four initial derivatives for boundedness. Prove necessity even when the secular terms oscillate and have infinitely many zeros.
4. Find a conserved quadratic expression involving only the last two state coordinates. Decide whether the full four-state system can have a conserved positive-definite quadratic form, and justify your answer.

Question 9 – Solution

Strategy. Expose an internal oscillator that resonantly forces the displacement oscillator.

Step 1: Reveal the oscillator cascade. The original equation gives

$$W' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} W.$$

The coordinate map is triangular with determinant one; its inverse is $(y, y', y'', y''') = (W_1, W_2, W_3 - W_1, W_4 - W_2)$. The last pair satisfies $W_3' = W_4$, $W_4' = -W_3$ independently.

Step 2: Solve the resonant forcing. We have $W_3 = A \cos t + B \sin t$ and $y'' + y = W_3$. A resonant particular solution is $(A/2)t \sin t - (B/2)t \cos t$. Enforcing the first two data gives

$$y = a \cos t + (b + B/2) \sin t + (A/2)t \sin t - (B/2)t \cos t.$$

It has $y''(0) = A - a$ and $y'''(0) = B - b$, checking all four data. Applying $D^2 + 1$ twice verifies the fourth-order equation.

Step 3: Classify bounded data. If $A = B = 0$, only bounded sine and cosine remain. If $(A, B) \neq (0, 0)$, choose a phase θ with $A \sin \theta - B \cos \theta = \sqrt{A^2 + B^2}$. Along $t_n = \theta + 2\pi n \geq 0$, the secular part grows as $t_n \sqrt{A^2 + B^2}/2$, while the other terms remain bounded. Consequently

$$y \text{ bounded} \iff y''(0) = -y(0), \quad y'''(0) = -y'(0).$$

The whole state is bounded exactly on this same two-dimensional subspace.

Step 4: Distinguish two kinds of conserved energy. The internal oscillator has

$$E_{\text{int}} = \frac{1}{2}(W_3^2 + W_4^2) = \frac{1}{2}(A^2 + B^2), \quad E'_{\text{int}} = W_3 W_4 - W_4 W_3 = 0.$$

This is not positive definite on the full state space. If a conserved positive-definite form $W^T P W$ existed, its smallest eigenvalue would bound $\|W(t)\|^2$ by a constant for every initial state. The unbounded solutions just constructed contradict that conclusion. Purely imaginary characteristic roots with repeated factors therefore do not imply boundedness of all trajectories.