

Question 8

The functions $1, t, e^t$ form a scalar basis for $y''' - y'' = 0$. For the derivative state $X = (y, y', y'')^T$, let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

A fundamental matrix need not equal the identity at the chosen base time.

Tasks

1. Form the fundamental matrix $F(t)$ from the given scalar basis and its derivatives. Check $F' = AF$ and compute its determinant.
2. Derive the transition matrix $\Phi(t, s) = F(t)F(s)^{-1}$ explicitly. Verify its normalization, composition law and determinant.
3. Use this transition matrix to solve $y''' - y'' = 1$ with $(y, y', y'')(1) = (0, 0, 0)$ for every real t . Treat the integral orientation when $t < 1$.
4. Explain why using $F(t)$ directly as the transition from time zero gives wrong initial data in general. Prove that replacing F by FC , for any constant invertible C , leaves the true transition matrix unchanged.

Question 8 – Solution

Strategy. Normalize a scalar-derived fundamental matrix at the actual base time before transporting states or forcing.

Step 1: Lift the scalar basis. The derivative columns give

$$F(t) = \begin{pmatrix} 1 & t & e^t \\ 0 & 1 & e^t \\ 0 & 0 & e^t \end{pmatrix}, \quad \det F = e^t \neq 0.$$

Differentiating the rows gives $F' = AF$, including the last row because $(e^t)' = e^t$. Thus F is fundamental on the entire real line.

Step 2: Normalize and compose. With $r = t - s$, direct multiplication gives

$$\Phi(t, s) = \begin{pmatrix} 1 & r & e^r - 1 - r \\ 0 & 1 & e^r - 1 \\ 0 & 0 & e^r \end{pmatrix}.$$

It equals I at $t = s$, and $\det \Phi = e^{t-s}$. The identity $F(t)F(s)^{-1}F(s)F(a)^{-1} = F(t)F(a)^{-1}$ proves $\Phi(t, s)\Phi(s, a) = \Phi(t, a)$ for all real base times.

Step 3: Transport the forcing in both directions. The zero initial vector at time one gives

$$X(t) = \int_1^t \Phi(t, v)e_3 dv, \quad y(t) = e^{t-1} - 1 - (t-1) - \frac{1}{2}(t-1)^2.$$

Indeed its first component integrates $e^{t-v} - 1 - (t-v)$. For $t < 1$ the integral is oriented: it is minus the integral from t to 1. Writing $r = t - 1$, we obtain $y' = e^r - 1 - r$, $y'' = e^r - 1$, $y''' = e^r$. These give the zero state at $t = 1$ and residual $y''' - y'' = 1$ on all of $\mathbb{R}$.

Step 4: Separate basis from transition. Here $F(0) \neq I$. For example, $F(t)e_3 = (e^t, e^t, e^t)^T$ starts at $(1, 1, 1)^T$, not at e_3 . The correct expression is $F(t)F(0)^{-1}e_3$. For any constant invertible C ,

$$(F(t)C)(F(s)C)^{-1} = F(t)CC^{-1}F(s)^{-1} = \Phi(t, s).$$

A basis change alters coordinates of solution columns, but not the transport of a specified physical state.