

Question 6

Let $D = d/dt$. Ordered differential factors act from right to left in

$$(D + 1)(D - t)(D - 1)y = 0, \quad (y, y', y'')(0) = (0, 0, 1).$$

Here $(D - t)f = f' - tf$. Multiplication by t does not commute with differentiation.

Tasks

1. Define $q_1 = y$, $q_2 = (D - 1)y$, $q_3 = (D - t)q_2$. Derive the first-order system for q and its initial vector.
2. Expand the scalar third-order equation and show that the change between (y, y', y'') and q is invertible for every real t .
3. Solve the triangular system by successive integrating factors. Give an explicit nested-integral formula for y and verify the initial data and equation through the state relations.
4. Compare the given operator with $(D - t)(D + 1)(D - 1)$. Compute their difference, and exhibit a function that solves the reordered equation but not the original one.

Question 6 – Solution

Strategy. Keep the factors in their stated order and regard intermediate derivatives as state coordinates.

Step 1: Form the ordered cascade. The definitions give

$$\boxed{q'_1 = q_1 + q_2, \quad q'_2 = tq_2 + q_3, \quad q'_3 = -q_3, \quad q(0) = (0, 0, 1)^T.}$$

In particular, $q_3 = y'' - (t+1)y' + ty$; using only y'' as this coordinate would lose the variable-coefficient terms.

Step 2: Expand and invert the coordinate map. Differentiating the expression for q_3 and adding q_3 yields

$$\boxed{y''' - ty'' - 2y' + (t+1)y = 0.}$$

The coordinate matrix is

$$q = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ t & -(t+1) & 1 \end{pmatrix} \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix}, \quad \det = 1.$$

Its inverse relations are $y = q_1$, $y' = q_1 + q_2$ and $y'' = q_1 + (t+1)q_2 + q_3$, so the conversion is globally reversible.

Step 3: Integrate in triangular order. Successive integrating factors give

$$q_3 = e^{-t}, \quad q_2 = e^{t^2/2} \int_0^t e^{-v-v^2/2} dv,$$

$$\boxed{y(t) = e^t \int_0^t e^{-u+u^2/2} \left(\int_0^u e^{-v-v^2/2} dv \right) du.}$$

These oriented integrals apply to every real t . Their derivatives give the three cascade equations. At zero the state is $(0, 0, 1)$, and the inverse relations give $(y, y', y'') = (0, 0, 1)$. Finally $q'_3 + q_3 = 0$ verifies the original ordered product directly.

Step 4: Quantify the order error. The commutator $(D+1)(D-t) - (D-t)(D+1)$ is multiplication by -1 . Thus the original operator minus the reordered one equals $-(D-1)$. The reordered operator is $D^3 - tD^2 - D + t$, whereas the original has the extra $-D + 1$. For $y = e^{-t}$ the reordered product vanishes, but the original gives $2e^{-t} \neq 0$. Reordering these factors changes the solution space.