

Question 5

Two coupled displacements satisfy

$$x'' + 2x - z = 0, \quad z'' + 2z - x = 0,$$

with $(x, x', z, z')(0) = (1, 0, 0, 0)$. Time is measured in units that make the coefficients dimensionless.

Tasks

1. Write a four-state first-order system using the order (x, x', z, z') . Eliminate z to obtain a fourth-order scalar equation for x and translate all four initial values.
2. Introduce $u = x + z$, $v = x - z$. Solve the resulting independent equations and recover x, z .
3. Prove that every solution of the scalar fourth-order equation reconstructs a unique solution of the coupled equations. Verify the prescribed solution and its initial data.
4. Derive a positive conserved energy, express it in u, v , and use it to prove boundedness. Sketch both displacements for $0 \leq t \leq 12$ and explain how two different frequencies appear.

Question 5 – Solution

Strategy. Compare the derivative-state, scalar-elimination and normal-mode descriptions of the same four-dimensional motion.

Step 1: Convert and transfer the initial state. For $X = (x, x', z, z')^T$,

$$X' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{pmatrix} X.$$

Since $z = x'' + 2x$, substitution into the second equation gives

$$x^{(4)} + 4x'' + 3x = 0, \quad (x, x', x'', x''')(0) = (1, 0, -2, 0).$$

Step 2: Solve the normal modes. Adding and subtracting give $u'' + u = 0$, $v'' + 3v = 0$, with initial displacements one and velocities zero. Thus

$$x = \frac{1}{2}(\cos t + \cos(\sqrt{3}t)), \quad z = \frac{1}{2}(\cos t - \cos(\sqrt{3}t)).$$

Step 3: Prove reversibility. For any scalar solution, define $z = x'' + 2x$. The first coupled equation is then an identity, and the second has residual $x^{(4)} + 4x'' + 3x = 0$. This formula also proves uniqueness of z . The displayed cosines satisfy the scalar factors $(D^2 + 1)(D^2 + 3)$ and give $z(0) = z'(0) = 0$ as required.

Step 4: Check energy and boundedness. Multiplying the equations by x', z' and adding proves conservation of

$$E = \frac{1}{2}(x'^2 + z'^2) + x^2 + z^2 - xz = \frac{1}{4}(u'^2 + u'^2 + v'^2 + 3v'^2) = 1.$$

Since $x^2 + z^2 - xz \geq (x^2 + z^2)/2$, both displacements are bounded; the explicit formulas sharpen this to $|x|, |z| \leq 1$. Each displacement mixes the frequencies 1 and $\sqrt{3}$.

