

Question 4

A smooth system on the entire real line is

$$x' = z, \quad z' = tw, \quad w' = 0.$$

Eliminating z, w appears to produce a third-order scalar equation for x . The elimination must be examined at $t = 0$.

Tasks

1. Eliminate z, w for $t \neq 0$. Explain why the resulting scalar equation is singular at the origin although the original system is regular there.
2. Solve the original system for arbitrary data $(x, z, w)(0) = (a, b, c)$. Determine which derivatives of x at zero encode these three constants.
3. Classify all $C^3(\mathbb{R})$ solutions of the scalar equation. Prove that each reconstructs a unique smooth system solution, explicitly treating the origin.
4. Decide whether $x(0), x'(0), x''(0)$ alone determine a unique scalar solution. Give a one-parameter counterexample if needed, and sketch three members sharing the zero derivative triple on $[-1, 1]$.

Question 4 – Solution

Strategy. Retain the state coordinate that division by t conceals at the singular point.

Step 1: Eliminate on the regular subintervals. Since $x'' = tw$, for $t \neq 0$ we have $w = x''/t$. Differentiating gives

$$tx''' - x'' = 0.$$

The leading coefficient vanishes at zero. This singularity came from the chosen scalar coordinate; the original system matrix is continuous everywhere.

Step 2: Solve before dividing. Integrating successively gives

$$w = c, \quad z = b + ct^2/2, \quad x = a + bt + ct^3/6.$$

Thus $(x, x', x'')(0) = (a, b, 0)$, while $x'''(0) = c$. The missing state coordinate is encoded in the third derivative, not in the usual second derivative.

Step 3: Check global equivalence. On either side of zero, $(x''/t)' = 0$, so $x = a_{\pm} + b_{\pm}t + c_{\pm}t^3/6$. Continuity of x, x' matches $a_{\pm}, b_{\pm}$, and continuity of x''' matches $c_{\pm}$. All C^3 solutions therefore have the single global polynomial form above. Reconstruct $z = x'$ and $w = x''/t$ off zero, setting $w(0) = x'''(0)$. These are smooth and satisfy all three system equations. At zero, the scalar equation itself requires $x''(0) = 0$.

Step 4: Explain the apparent nonuniqueness. The family $x = ct^3/6$ has $(x, x', x'')(0) = (0, 0, 0)$ for every c . It corresponds to distinct system data $(0, 0, c)$, so it does not contradict uniqueness for the regular system. The plotted members use $c = -6, 0, 6$.

