

Question 2

A scalar measurement of a three-state system is

$$X' = AX, \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{pmatrix}, \quad y = X_1 + 2X_2.$$

The measured initial derivatives are $y(0) = 1$, $y'(0) = 0$, $y''(0) = -12$. All coefficients are constant and $t \in \mathbb{R}$.

Tasks

1. Form the matrix O such that $(y, y', y'')^T = OX$. Determine whether the three measured derivatives uniquely determine the state.
2. Derive a scalar third-order equation for y and explicit reconstruction formulas for all three state components in terms of y, y', y'' .
3. Recover $X(0)$ and find the measured response $y(t)$ explicitly.
4. Prove that every solution of your scalar equation reconstructs a solution of the original system. Verify the recovered trajectory and explain why differentiating the measurement has lost no state information.

Question 2 – Solution

Strategy. Invert the map from state coordinates to the measurement's derivative coordinates.

Step 1: Test the measurement map. With $c = (1, 2, 0)$, the rows of O are c, cA, cA^2 . Hence

$$O = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ -12 & -22 & -11 \end{pmatrix}, \quad \det O = -15 \neq 0.$$

The derivative triple determines the state uniquely.

Step 2: Convert and reconstruct. The identity $A^3 + 6A^2 + 11A + 6I = 0$ gives

$$y''' + 6y'' + 11y' + 6y = 0.$$

Solving $OX = (y, y', y'')^T$ gives

$$\begin{aligned} X_1 &= (-33y - 22y' - 4y'')/15, \\ X_2 &= (24y + 11y' + 2y'')/15, \\ X_3 &= (-12y + 2y' - y'')/15. \end{aligned}$$

Step 3: Solve the recovered IVP. The measured data give $X(0) = (1, 0, 0)^T$. The scalar roots are $-1, -2, -3$; imposing the measured derivative triple yields

$$y = -3e^{-t} + 9e^{-2t} - 5e^{-3t}.$$

The state is $X = (f, f', f'')^T$, where $f = 3e^{-t} - 3e^{-2t} + e^{-3t}$. In particular, $f + 2f' = y$ and $(f, f', f'')(0) = (1, 0, 0)$.

Step 4: Prove full equivalence. If $Z = (y, y', y'')^T$, then $Z' = BZ$ with last row of B equal to $(-6, -11, -6)$. Direct multiplication gives $OA = BO$, so for any scalar solution, $X = O^{-1}Z$ satisfies $X' = O^{-1}BZ = AX$. Its measurement is y by the first row of O . Conversely every state solution produces this scalar equation. The recovered f satisfies $f''' + 6f'' + 11f' + 6f = 0$, checking the original last row directly. Invertibility of O proves that no hidden state was discarded.