

Question 10

For $\varepsilon \geq 0$, let h_ε be the causal impulse response of

$$(D + 1)^2(D + 1 + \varepsilon)h_\varepsilon = \delta(t),$$

with zero prehistory. The impulse is at the origin; use the causal transform convention $\mathcal{L}\delta(t) = 1$. Ordinary formulas below describe $t \geq 0$.

Tasks

1. Find the transform and invert it for $\varepsilon > 0$ by partial fractions in $z = s + 1$.
2. Derive a convolution formula with a nonnegative integrand. Use it to obtain h_0 and prove $h_\varepsilon \rightarrow h_0$ without subtracting divergent coefficients.
3. Prove two-sided bounds for $h_\varepsilon(t)$ and an explicit bound of the form $\sup_{t \geq 0} |h_\varepsilon(t) - h_0(t)| \leq C\varepsilon$ with a numerical constant C .
4. Explain the change in pole multiplicities and verify the four right-hand initial derivatives required by a unit impulse. Discuss reliable evaluation for small εt , and sketch the kernels for $\varepsilon = 0, 1/2, 2$.

Question 10 – Solution

Strategy. Use partial fractions for a closed form and convolution for a stable coalescing-pole limit.

Step 1: Invert distinct repeated poles. For $\varepsilon > 0$, $H_\varepsilon = 1/[z^2(z + \varepsilon)^2]$ and

$$H_\varepsilon = \frac{1}{\varepsilon^2} \left(\frac{1}{z^2} + \frac{1}{(z + \varepsilon)^2} \right) + \frac{2}{\varepsilon^3} \left(\frac{1}{z + \varepsilon} - \frac{1}{z} \right).$$

Hence $h_\varepsilon = e^{-t} \{t(1 + e^{-\varepsilon t})/\varepsilon^2 - 2(1 - e^{-\varepsilon t})/\varepsilon^3\}$.

Step 2: Take the limit through convolution. Convolving te^{-t} with $te^{-(1+\varepsilon)t}$ gives

$$h_\varepsilon(t) = e^{-t} \int_0^t v(t-v)e^{-\varepsilon v} dv, \quad h_0(t) = \frac{t^3 e^{-t}}{6}.$$

For fixed t , dominated convergence justifies the limit, including $t = 0$.

Step 3: Prove global error bounds. Bounding the integrand and using $1 - e^{-\varepsilon v} \leq \varepsilon v$ yields

$$e^{-(1+\varepsilon)t} \frac{t^3}{6} \leq h_\varepsilon(t) \leq e^{-t} \frac{t^3}{6}, \quad 0 \leq h_0 - h_\varepsilon \leq \frac{\varepsilon t^4 e^{-t}}{12}.$$

The maximum of $t^4 e^{-t}$ is $256e^{-4}$ at $t = 4$. Thus $\sup_{t \geq 0} |h_\varepsilon - h_0| \leq 64\varepsilon/(3e^4)$.

Step 4: Check poles, impulses and evaluation. Two double poles merge into a quadruple pole at -1 ; absolute convergence holds for $\operatorname{Re} s > -1$. Convolution gives $h_\varepsilon = t^3/6 + O(t^4)$, so its right-hand jet is $(0, 0, 0, 1)$, producing exactly a unit δ . For small εt , use the integral or its exponential series; the closed form subtracts large nearly equal terms and can lose significant digits.

