

Question 9

An experiment supplies the exact transform

$$Y(s) = \frac{s^4 + 3s^3 + 7s^2 + 11s + 18}{(s+3)(s^2+1)(s^2+4)}$$

of a solution to

$$y^{(4)} + 5y'' + 4y = Ae^{-3t}, \quad t \geq 0.$$

The forcing amplitude A and all four initial values are unknown real numbers.

Tasks

1. Recover A and $(y, y', y'', y''')(0)$ by polynomial division of $(s^2+1)(s^2+4)Y$. Prove that this recovery is unique.
2. Invert Y completely using real sine, cosine and exponential terms.
3. Verify the recovered forcing and all four initial values directly from your answer, and state an absolute-convergence half-plane.
4. Find the monic constant-coefficient homogeneous differential operator of least order that annihilates this particular response. Explain why its order differs from that of the forced equation, and prove minimality.

Question 9 – Solution

Strategy. Separate the polynomial encoding of initial data from the proper rational forcing term.

Step 1: Recover data uniquely. Write $P(s) = s^4 + 5s^2 + 4$. Polynomial division gives

$$P(s)Y = s^3 + 7s - 10 + \frac{48}{s+3}.$$

For initial values $c_j = y^{(j)}(0)$, transformation of the equation gives

$$PY = c_0s^3 + c_1s^2 + (c_2 + 5c_0)s + (c_3 + 5c_1) + \frac{A}{s+3}.$$

Thus $\boxed{A = 48, \quad (c_0, c_1, c_2, c_3) = (1, 0, 2, -10)}$. Polynomial division has a unique polynomial part and remainder; the displayed triangular relation then determines every c_j uniquely.

Step 2: Resolve the real modes. Real partial fractions yield

$$Y = \frac{(2/5)s + 22/15}{s^2 + 1} + \frac{(3/13)s - 14/39}{s^2 + 4} + \frac{24/65}{s + 3},$$

so

$$\boxed{y = \frac{2}{5} \cos t + \frac{22}{15} \sin t + \frac{3}{13} \cos(2t) - \frac{7}{39} \sin(2t) + \frac{24}{65} e^{-3t}.$$

Step 3: Check the recovered experiment. The oscillatory terms are annihilated by $P(D)$ and $P(-3) = 130$, so the exponential contributes $130(24/65)e^{-3t} = 48e^{-3t}$. At zero the four derivatives are

$$\begin{aligned} y(0) &= 2/5 + 3/13 + 24/65 = 1, \\ y'(0) &= 22/15 - 14/39 - 72/65 = 0, \\ y''(0) &= -2/5 - 12/13 + 216/65 = 2, \\ y'''(0) &= -22/15 + 56/39 - 648/65 = -10. \end{aligned}$$

Absolute convergence holds for $\operatorname{Re} s > 0$.

Step 4: Prove the minimum homogeneous order. All five complex modes at $i, -i, 2i, -2i, -3$ have nonzero coefficients. Their linear independence forces every annihilating polynomial to vanish at all five distinct roots. Hence the least monic operator is $\boxed{(D+3)(D^2+1)(D^2+4)}$, of order five. The fourth-order equation leaves the exponential as a nonzero forcing; annihilating that forcing adds the fifth factor.