

Question 8

A stable third-order filter receives identical unit impulses at times $T, 2T, 3T, \dots$, where $T > 0$:

$$(D + 1)^3 y = \sum_{n=1}^{\infty} \delta(t - nT).$$

Assume zero prehistory and zero initial state. Interpret the equation in the sense of distributions; there is no impulse at $t = 0$.

Tasks

1. Find $Y(s)$, stating a convergence half-plane. Give the exact finite-sum response at any finite time.
2. For $t = NT + x$ with $0 \leq x < T$, find the limiting periodic profile as $N \rightarrow \infty$. Sum it explicitly using $q = e^{-T}$.
3. Determine the continuity and derivative jumps across each impulse, including the join in the limiting profile. Compute the mean of the limiting profile over a period.
4. Decide whether $y(t)$ has a pointwise limit as $t \rightarrow \infty$. Explain the failure or validity of a final-value argument, and sketch the transient and limiting profile for $T = 2$.

Question 8 – Solution

Strategy. Sum delayed impulse kernels and distinguish a periodic limiting profile from a constant final value.

Step 1: Sum the transformed impulses. For $\operatorname{Re} s > 0$,

$$Y = \frac{e^{-sT}}{(1 - e^{-sT})(s + 1)^3}, \quad y(t) = \sum_{1 \leq n \leq \lfloor t/T \rfloor} h(t - nT), \quad h(v) = \frac{1}{2}v^2e^{-v}.$$

The sum is empty before T ; including the current impulse adds $h(0) = 0$.

Step 2: Sum the limiting profile. At $t = NT + x$ the sum is $\sum_{k=0}^{N-1} h(x + kT)$. Geometric-series identities give

$$p(x) = \frac{e^{-x}}{2} \left[\frac{x^2}{1 - q} + \frac{2xTq}{(1 - q)^2} + \frac{T^2q(1 + q)}{(1 - q)^3} \right], \quad q = e^{-T}.$$

The omitted tail is uniformly $O(N^2q^N)$ for $0 \leq x \leq T$ and fixed T , so $y(NT + x) \rightarrow p(x)$ uniformly on a period.

Step 3: Check joins and mean. Since $(h, h', h'')(0) = (0, 0, 1)$, y, y' are continuous and y'' jumps upward by one. Reindexing the convergent profile series gives the same periodic joins: $p(0) = p(T-)$, $p'(0) = p'(T-)$, $p''(0) - p''(T-) = 1$. Also $\int_0^T p(x) dx = \int_0^\infty h(v) dv = 1$, so its mean is $1/T$.

Step 4: Reject a constant final value. The profile is nonconstant: on each open gap $(D+1)^3p = 0$, so a constant profile would be zero, contradicting its integral. Distinct phases therefore have distinct subsequential limits. Poles at $s = 2\pi ik/T$, $k \neq 0$, remain in sY ; the formal limit $\lim_{s \rightarrow 0} sY = 1/T$ is a mean, not a final value.

